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APPLICATION OF HIGH-SPEED COMPUTERS TO THE SOLUTION OF THE RESTRICTED THREE-BODY PROBLEM BY THE HILL-BROWN METHOD; PART I: CALCULATION OF THE RIGHT-HAND SIDES OF THE EQUATIONS IN INHOMOGENEOUS FORM

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V. A. Shor

ABSTRACT

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The use of high-speed electronic computers is investigated for solving the restricted three-body problem by the Hill-Brown method. The logic diagram is given for a routine to compute terms on the right-hand sides of the inhomogeneous equations generated by the derivatives of the perturbation function. The same program is used to compute the terms produced by series expansion of $u\zeta^{-1}/r^3$, $z\sqrt{-1}/r^3$.

author

During the last ten years, the Institute of Theoretical Astronomy, Academy of Sciences of the USSR has undertaken several attemps to utilize the lunar method of Hill and Brown for the formulation of analytical theories of certain irregular satellites of Jupiter (Proskurin, ref. 6 and 7, Tokmalayeva, ref. 9). Theoretical justification was found for applying the lunar method at least to some of the satellites of Jupiter. The obstacles standing in the way of wide-spread application of this method to satellite systems are not so much theoretical as of a purely practical nature; for machine calculations, the formulation of a theory comparable in accuracy with observations requires several years of concentrated effort, even for satellites with fairly modest orbit parameters.

*Numbers in the margin indicate pagination in the original foreign text.

These difficulties are compounded by the fact that the parameter m, which enters into the theory in numerical form, is not known with sufficient accuracy for a number of satellites, hence a theory, even it it were formulated, could not be regarded as complete; more precise knowledge of the orbit parameters requires recomputation of at least some of the inequalities. However, even at this moment we are sorely in need of a theory, because the purely literal theory of Delauney is either superficially applicable for some of the distant satellites of Jupiter (satellites VI, VII, X) or generally inapplicable (satellite VIII).

We should also add that the application of the Delauney theory just to account for the perturbations requires rather numerous and tedious computations, to the extent that it would be encouraging if even some of the necessary labor could be mechanized.

The above has justified investigating the possibility of using electronic computers to automate the computational process in the Hill-Brown method. The possibility and desirability of mechanizing the separate states of the compu- /640
tations in this method were conceptually demonstrated by Brown himself some years ago (1938, ref. 15). More recently, a number of papers have been published in which the problem of implementing classical computational methods on electronic equipment has been coped with successfully (refs. 8 and 16).

Of course, in implementing methods that were created in application to the computational devices of the last century, we cannot utilize to the fullest extent the capabilities of electronic computers; we require the creation of new methods basically oriented toward modern techniques. However, by using the old method with a well developed solution algorithm, we achieve our goal more quickly, and the first results that have in fact been obtained in the indicated papers demonstrate the wisdom of this approach to solution of a number of problems in celestial mechanics.

A number of features of the Hill-Brown method are clearly amenable to the workable solution of the stated problem.

About two thirds of all the work involved in this method calls for the multiplication of series, a process which yields very simply to mechanization. The entire sequence of operations that must be executed in order to determine the coefficients of the inequalities with a certain characteristic forms a computational cycle, which is iterated as many times as there are characteristics to be taken into account. Since there can be as many as a hundred or more, we see at once the utility of the method; the cyclical nature of the operation leaves no doubt. Moreover, the analysis of one cycle shows that it breaks down into a number of stages, within which the computations are of an exceedingly uniform, repetitive, and essentially elementary nature. These stages include selection of the necessary terms from the expansion of the perturbation function, the multiplication of inequalities, the solution of equations by the method of successive approximations.

On the other hand, it is important to realize that the lunar method of Hill and Brown is recognized as a method for solving the restricted three-body problem, and only this fundamental problem can be fitted to a unified scheme of solution. To account for other effects in satellite motion requires a particular approach and a special computational program.

We note one other feature of the Brown method which is very useful in manual calculation but which clearly cannot be used in computer calculations. In calculating the coefficients of the inequalities manually, we are often able to make use of the products of series which appear as intermediate results in determining lower-order inequalities.

The peculiar nature of computations on electronic equipment is such that it is simpler and faster to reproduce these products by repeated computation than to store them and later extract them from memory. In this case, however, the computations lead to a progressive increase in the time spent in determining the coefficients as the orders of the inequalities is increased.

Brown (ref. 15) developed the notion that utilizing the equations of motion in a rectilinear coordinate system rotating with an angular velocity equal to the mean motion of the moon should greatly facilitate the procedure of determining the unknown coefficients of the inequalities.

However, the writing of an algorithm to solve these equations represents in itself a rather laborious task. If it proves worthwhile in practice to implement the lunar method on computers, then at that time the problem of changing the coordinate system can be taken up.

In formulating the theory of lunar motion, Brown employed the equations of motion in the so-called homogeneous and inhomogeneous forms.

The solution of the equations in inhomogeneous form involves less effort (this applies to the determination of the coefficients of the several low order inequalities), but with small divisors present the accuracy of the coefficients turns out to be insufficient in a number of instances. In these cases, the precision of the coefficients is increased by means of the equations in homogeneous form. The coefficients of the fifth- and sixth-order inequalities have been /641 computed on the basis of the equations in homogeneous form exclusively. In computer calculations, the use of both forms of equations is undesirable, since in this case it becomes necessary to have two different programs. Consequently, at the expense of a certain increase in the number of computations, it is necessary to restrict to the homogeneous equations.

As our work progressed, however, it was decided first of all to explore the possibility of completely automating the computational process using the equations in inhomogeneous form. This was all the more justified in that the computation of the right-hand members of the equations in homogeneous form can be carried out to a considerable extent in the same general scheme as for the inhomogeneous form of the equations. The equations are solved in either case by the method of successive approximations, which makes it possible to carry over a number of the techniques used in solving the equations in inhomogeneous form to solution of the equations in homogeneous form. Furthermore, subsequent experience in the operation of the program will uncover its weak points and permit the exclusion of errors in solving the more difficult problem of automating the solution of the equations in homogeneous form.

In the present article, we present the results pertaining to computation of the right-hand sides of the equations in inhomogeneous form. In so doing, we will steer away from an exhaustive description of the program, treating instead only the principal stages of the computational process and the techniques used to solve some of the problems that arise in programming. The proposed computational scheme, of course, is not the only possible one; of several investigated modifications, we have chosen the one which, in our opinion, was better suited than the others to the capabilities of the Strela computer; the latter has an internal memory space of 2047 cells plus a magnetic tape memory device; it has a computation rate up to 2000 operations per second.

Aiming first of all at the formulation of theories to account for some of the irregular satellites of Jupiter (VI, VII, X), in the expansions of the derivatives of the perturbation function terms we have retained terms of order no higher than $\alpha^2 e'$, $\alpha e'^2$, e'^3, $\alpha^2 z^2$, since for these satellites, as shown by

Tokmalayeva's estimates (ref. 9), we can neglect inequalities having factors of lesser magnitude, provided only the error does not have to be better than 0.''1 in geocentric longitude. However, the principle by which the program is formulated does not at all depend on the number of terms retained in the expansions.

In the interest of saving the reader constant referral to primary sources, we will briefly review the basic postulates of the Hill-Brown theory in the first two sections.

The work was carried out under the direction of Prof. N. S. Yakhontovaya, to whom the author expresses his deep appreciation.

1. EQUATIONS OF MOTION AND FORM OF SOLUTION

Let x, y, z be the rectilinear coordinates of a satellite relative to a moving system of axes with origin at the point J (Jupiter) and base plane coinciding with the plane of the sun's orbit. The x-axis is directed toward the center of the sun, forming a right-handed reference frame with the y- and z-axes. As assumed, the motion of the sun about the center of inertia of Jupiter and the satellite is executed in a fixed plane in an elliptical orbit.

We let

$$u=x+\sqrt{-1}\,y,\quad \zeta=\exp(n-n')(t-t_0)\sqrt{-1},\quad \varkappa=\frac{k^2I}{(n-n')^2},$$

$$s=x-\sqrt{-1}\,y,\quad D=\frac{1}{(n-n')\sqrt{-1}}\cdot\frac{d}{dt}=\zeta\frac{d}{d\zeta},\quad m=\frac{n'}{n-n'},$$

where n, n are the mean motions of the satellite and the sun, I is the mass of Jupiter. The mass of the satellite is assumed equal to zero.

Then the equations of motion of the satellite will have the form (ref. 13, /642 1899, p. 53)

$$\left.\begin{aligned}(D+m)^2u+\frac{1}{2}m^2u+\frac{3}{2}m^2s-\frac{\varkappa u}{(us+z^2)^{3/2}}&=-\frac{\partial\Omega}{\partial s},\\(D-m)^2s+\frac{1}{2}m^2s+\frac{3}{2}m^2u-\frac{\varkappa s}{(us+z^2)^{3/2}}&=-\frac{\partial\Omega}{\partial u},\\(D^2-m^2)z-\frac{\varkappa z}{(us+z^2)^{3/2}}&=-\frac{1}{2}\frac{\partial\Omega}{\partial z}.\end{aligned}\right\}\qquad(1)$$

Here Ω is the perturbation function of the restricted three-body problem:

$$\Omega = m^2\left[\frac{3}{4}(u^2 a_2 + s^2\bar{a}_2) + \frac{1}{2}usb_2 - z^2 b_2\right] +$$
$$+ \frac{m^2}{a'}\left[\frac{5}{8}(u^3 a_3 + s^3\bar{a}_3) + \frac{3}{8}(u^2 s c_3 + us^2\bar{c}_3) - \frac{3}{2}uz^2 c_3 - \frac{3}{2}sz^2 \bar{c}_3\right] +$$
$$+ \frac{m^2}{a'^2}\left[\frac{35}{64}(u^4 a_4 + s^4\bar{a}_4) + \frac{5}{16}(u^3 s c_4 + us^3\bar{c}_4) + \frac{9}{32}u^2 s^2 b_4 - \right.$$
$$\left. - z^2\left(\frac{15}{8}u^2 c_4 + \frac{15}{8}s^2\bar{c}_4 + \frac{9}{4}usb_4\right)\right], \qquad (2)$$

where

$$\left.\begin{aligned}
a_2 &= e'\left(-\frac{1}{2}\zeta^m + \frac{3}{2}\zeta^{-m}\right) + e'^2\left(-\frac{5}{2} + \frac{17}{2}\zeta^{-2m}\right) + e'^3\left(\frac{1}{16}\zeta^m - \frac{123}{16}\zeta^{-m} + \right.\\
&\qquad \left. + \frac{1}{48}\zeta^{3m} + \frac{845}{48}\zeta^{-3m}\right),\\
b_2 &= \frac{3}{2}e'(\zeta^m + \zeta^{-m}) + e'^2\left(\frac{3}{2} + \frac{9}{4}\zeta^{2m} + \frac{9}{4}\zeta^{-2m}\right) + e'^3\left(\frac{27}{16}\zeta^m + \frac{27}{16}\zeta^{-m} + \right.\\
&\qquad \left. + \frac{53}{16}\zeta^{3m} + \frac{53}{16}\zeta^{-3m}\right),\\
a_3 &= 1 + e'(-\zeta^m + 5\zeta^{-m}) + e'^2\left(-6 + \frac{1}{8}\zeta^{2m} + \frac{127}{8}\zeta^{-2m}\right),\\
c_3 &= 1 + e'(\zeta^m + 3\zeta^{-m}) + e'^2\left(2 + \frac{11}{8}\zeta^{2m} + \frac{53}{8}\zeta^{-2m}\right),\\
a_4 &= 1 + e'\left(-\frac{3}{2}\zeta^m + \frac{13}{2}\zeta^{-m}\right),\\
c_4 &= 1 + e'\left(\frac{1}{2}\zeta^m + \frac{9}{2}\zeta^{-m}\right),\\
b_4 &= 1 + e'\left(\frac{5}{2}\zeta^m + \frac{5}{2}\zeta^{-m}\right).
\end{aligned}\right\} \qquad (3)$$

The values of $\bar{a}_2$, $\bar{a}_3$,... are obtained from a_2, a_3,... by substitution of ζ^{-1} for ζ.

The solutions x, y, z of these equations can be formally represented as trigonometric series in multiples of the four arguments

$$(n-n')(t-t_0), \qquad c(n-n')(t-t_1),$$
$$m(n-n')(t-t_2), \qquad g(n-n')(t-t_3),$$

for which $t_0 = -\frac{\varepsilon_0 - \varepsilon_0'}{n-n'}$, $t_1 = -\frac{\varepsilon_0 - \pi_0}{c(n-n')}$, $t_2 = -\frac{\varepsilon_0' - \pi_0'}{n'}$, $t_3 = -\frac{\varepsilon_0 - \Omega_0}{g(n-n')}$ are the difference in mean longitudes of the satellite and sun, mean anomaly of the satellite, mean anomaly of the sun, and mean latitude argument of the satellite, respectively (in these formulas, $\epsilon_0, \pi_0, \Omega_0$ are the mean longitude, perijove longitude, and nodal longitude of the satellite in the initial epoch, ϵ_0', π_0' are the longitude of the sun and the longitude of its pericenter in the same epoch, the quantities c and g remain to be determined). /643

The coefficients of the trigonometric series, in turn, are expanded in series ascending powers of **e**, e', **k**, and $\alpha = a/a'$, where **e, k, a** are parameters denoting the eccentricity, inclination, and semimajor axis of the satellite orbit, e' and a' are the eccentricity and semimajor axis of the sun's orbit.

Going over to complex variables, it may be stated that the solution $u\zeta^{-1}$, $s\zeta$, $z\sqrt{-1}$, correct to constant factors that are nonessential to the determination of the unknown coefficients $\lambda_{i,\tau}$, $\lambda_{i,-\tau}$, can be represented as the sum of inequalities, each of which has the form

$$\mathbf{a}\lambda \sum_i [\lambda_{i,\tau}\zeta^{2i+\tau} + \lambda_{i,-\tau}[illegible]^{2i-\tau}], \quad [illegible] = \mathbf{e}^{i_1} e'^{i_2} \mathbf{k}^{i_3} \alpha^{i_4}. \tag{4}$$

In this expression, λ is the characteristic of the inequality whose order is equal to $i_1 + i_2 + i_3 + i_4$. We can speak of the order of the inequality in the same sense. The set of all inequalities with the characteristic λ in the expansion of any coordinate will be called the complex of inequalities with that characteristic and will be denoted by $u_\lambda\zeta^{-1}$, $s_\lambda\zeta$, $z_\lambda\sqrt{-1}$, respectively. The values that $\pm\tau$ can assume for the separate inequalities of a given complex can be determined from the equation (ref. 14).

$$\tau = (i_1 - 2i_1')c + (i_2 - [illegible] - 2i_3')g \quad [illegible] i' = 0, 1, 2, \ldots [illegible] \tag{5}$$

In the characteristics of the inequalities of the coordinates u and s, the index i_3 is even, in the characteristics of the inequalities of the coordinate z the index i_3 id odd.

An expression of the type $\lambda_{i,\tau}\zeta^{2i+\tau}$ is called a term of the expansion with argument $2i + \tau$ (occasionally, for brevity, the term "argument" will be applied to the quantity τ).

The set of terms for which the quantity τ remains invariant forms a group. The index i for the separate terms of the group assumes those of the values 0, ±1/2, ±1, ±3/2,... for which the numbers 2i and i_4 are identically even or odd.

For the coefficients of the inequalities of the coordinate z, we have the relation

$$\lambda_{-i,\tau} = -\lambda_{i,-\tau}. \tag{6}$$

2. METHOD OF DETERMINING THE UNKNOWN COEFFICIENTS

For determining the unknown coefficients of the inequalities, the method of undetermined coefficients is used, where as a first approximation the coefficients of the zero-order inequalities are determined, followed by the coefficients of the first-order inequalities, second-order, and so forth.

Let the coefficients of the inequalities up to and including order n - 1 be already determined and let it be required to find the coefficients of the nth-order inequalities with characteristic λ. It follows from the foregoing section that each coordinate can be represented as the sum of complexes of inequalities:

$$\mu\zeta^{-1} = u_0\zeta + \sum u_\lambda \zeta^{-1}, \qquad s\zeta = s_0\zeta + \sum s_\mu \zeta, \qquad z\sqrt{-1} = \sum z_\mu \sqrt{-1},$$

where the characteristic μ takes on all possible values for a given coordinate except $\mu = 0$. Substituting these expressions into the equations of motion, we equate terms containing the factor λ on the right- and left-hand sides of the equations. This leads to the following relations (ref. 13, 1899): /644

$\zeta^{-1}(D + m)^2 u_\lambda + M u_\lambda \zeta^{-1} + N s_\lambda \zeta$ is equal to the sum of terms with characteristic λ

in the expression

$$
\begin{aligned}
&-\zeta^{-1}(D^2+2mD)\sum u_\mu-\frac{\partial\Omega}{\partial s}\zeta^{-1}+\frac{1}{\mathbf{a}}\left[\frac{3}{8}\bar{P}\left(\sum u_\mu\zeta^{-1}\right)^2+\frac{15}{8}Q\left(\sum s_\mu\zeta\right)^2+\right.\\
&+\frac{3}{4}P\sum u_\mu\zeta^{-1}\cdot\sum s_\mu\zeta+\left.\frac{3}{2}P\left(\sum z_\mu\sqrt{-1}\right)^2\right]-\frac{1}{\mathbf{a}^2}\left[\frac{5}{16}\bar{R}\left(\sum u_\mu\zeta^{-1}\right)^3+\right.\\
&+\frac{35}{16}T\left(\sum s_\mu\zeta\right)^3+\frac{9}{16}S\left(\sum u_\mu\zeta^{-1}\right)^2\sum s_\mu\zeta+\frac{15}{16}R\left(\sum s_\mu\zeta\right)^2\sum u_\mu\zeta^{-1}+\\
&+\frac{9}{4}S\left(\sum z_\mu\sqrt{-1}\right)^2\sum u_\mu\zeta^{-1}+[\text{illegible}]\left(\sum z_\mu\sqrt{-1}\right)^2 [\text{illegible}]\, s_\mu\zeta\left.\right]+\ldots
\end{aligned}
\tag{7}
$$

$D^2 z_\lambda\sqrt{-1} - 2Mz_\lambda\sqrt{-1}$ is equal to the sum of terms with characteristic λ in the expression

$$
\begin{aligned}
&-D^2\sum z_\mu\sqrt{-1}-\frac{1}{2}\frac{\partial\Omega}{\partial z}\sqrt{-1}-\frac{3}{2}\frac{\sum z_\mu\sqrt{-1}}{\mathbf{a}}\left[\bar{P}\sum u_\mu\zeta^{-1}+P\sum s_\mu\zeta\right]+\\
&+\frac{\sum z_\mu\sqrt{-1}}{\mathbf{a}^2}\left[\frac{15}{8}\bar{R}\left(\sum u_\mu\zeta^{-1}\right)^2+\frac{15}{8}R\left(\sum s_\mu\zeta\right)^2+\frac{9}{4}S\sum u_\mu\zeta^{-1}\cdot\sum s_\mu\zeta\right]+\frac{3}{2}S\frac{\left[\sum z_\mu\sqrt{-1}\right]^3}{\mathbf{a}^2}+\ldots,
\end{aligned}
$$

where

$$
M=\sum_i M_i\zeta^{2i}=[\text{illegible}]\,m^2+\frac{1}{2}\frac{\varkappa}{\rho_0^3},\qquad N=\sum_i N_i\zeta^{2i}=\frac{3}{2}m^2\zeta^{-2}+\frac{3}{2}\frac{\varkappa u_0^2\zeta^{-2}}{\rho_0^5},
$$

$$
P=\sum_i P_i\zeta^{2i}=\mathbf{a}\frac{\varkappa u_0\zeta^{-1}}{\rho_0^5},\qquad S=\sum_i S_i\zeta^{2i}=\mathbf{a}^2\frac{\varkappa}{\rho_0^5},
$$

$$
Q=\sum_i Q_i\zeta^{2i}=[\text{illegible}]\frac{\varkappa u_0^3\zeta^{-3}}{\rho_0^{[\text{illegible}]}},\qquad T=\sum_i T_i\zeta^{2i}=\mathbf{a}^{[\text{illegible}]}\frac{[\text{illegible}]\zeta^{-4}}{\rho_0^{[\text{illegible}]}},
$$

$$
R=\sum_i R_i\zeta^{2i}=\mathbf{a}^2\frac{\varkappa u_0^2\zeta^{-2}}{\rho_0^7}\qquad \rho_0^2=u_0\zeta^{-1}\cdot s_0\zeta.
$$

The quantities $\bar{P}$, $\bar{R}$,... are obtained from P, R,... by substitution of ζ^{-1} for ζ.

On the left-hand sides of equations (7) and (8) are enumerated all the terms containing the unknowns u_λ, s_λ, and z_λ. It is sufficient to assume that on the right-hand sides of these equations μ acquires characteristics of order no higher than n - 1, since higher-order inequalities could not produce terms therein with the factor λ.

condition yields a system comprising an infinite number of linear equations with an infinite number of unknowns:

$$\left.\begin{aligned} (2i+\tau+1+m)^2\lambda_{i,\tau}+\sum_j M_j\lambda_{i-j,\tau}+\sum_j N_j\lambda_{j-i,-\tau}&=A_{i,\tau},\\ (2i+\tau-1-m)^2\lambda_{-i,-\tau}+\sum_j M_j\lambda_{-i-j,-\tau}+\sum_j N_j\lambda_{j+i,\tau}&=A_{-i,-\tau},\end{aligned}\right\} \tag{10}$$

where j = 0, ±1, ±2,..., 2i = 0, ±1, ±2,..., $2i - i_4$ are always even.

The quantities $A_{i,\tau}$, $A_{i,-\tau}$, M_j, N_j decay rapidly with increasing $|i|$ and $|j|$, and this means that we can limit to a finite number of equations, which are solved simply by the method of successive approximations.

After rejecting the required terms, equation (8) assumes the form

$$D^2 z_\lambda\sqrt{-1}-2 z_\lambda\sqrt{-1}=a_\lambda\sqrt{-1}\sum A_{i,\tau}(\zeta^{2i+\tau}-\zeta^{-2i-\tau}). \tag{9a}$$

To determine the coefficients $\lambda_{i,\tau}$ of the inequalities of the coordinate z, each of which has the form

$$z_{\lambda,\tau}\sqrt{-1}=a\sum_i\lambda_{i,\tau}(\zeta^{2i+\tau}-\zeta^{-2i-\tau}),$$

we obtain the system

$$(2i+\tau)^2\lambda_{i,\tau}-2\sum_j M_j\lambda_{-j,\tau}=A_{i,\tau}, \tag{11}$$

where j = 0, ±1, ±2,..., 2i = 0, ±1, ±2,..., $2i - i_4$ are even, which is also solved by the method of successive approximations.

In the case of inequalities with characteristics **e** and **k** ($\tau = c$ and $\tau = g$), the systems (10) and (11) are homogeneous and can have nontrivial solutions only in the case when their determinants go to zero. This fact can be utilized to determine the unknowns c_0 and g_0. At the present time, the values of c_0 and g_0 which make the determinant go to zero can be obtained by means of the series

The presence of the expressions

$$-\zeta^{-1}(D^2 - 2mD)\sum u_\mu, \qquad -D^2\sum z_\mu\sqrt{-1}$$

on the right-hand sides of equations (7) and (8) is attributable to the fact that the quantities c and g are series in even powers of the fundamental parameters:

$$c = c_0 + e^2c_{e^2} + e'^2c_{e'^2} + \alpha^2c_{\alpha^2} + k^2c_{k^2} + e^4c_{e^4} + e^2e'^2c_{e^2e'^2} + \ldots,$$
$$g = g_0 + e^2g_{e^2} + e'^2g_{e'^2} + \alpha^2g_{\alpha^2} + k^2g_{k^2} + e^4g_{e^4} + e^2e'^2g_{e^2e'^2} + \ldots$$

In differentiating inequalities of lower order than n, terms of order n with the characteristic λ can therefore occur.

After the necessary terms have been eliminated from the right-hand side, equation (7) assumes the form

$$\zeta^{-1}(D+m)^2u_\lambda + Mu_\lambda\zeta^{-1} + Ns_\lambda\zeta = a\lambda\sum_{i,\tau}[A_{i,\tau}\zeta^{2i+\tau} + A_{i,-\tau}\zeta^{2i-\tau}]. \quad (9)$$

On the right-hand side of (9) we separate out two groups of terms with /645
arguments of the form 2i +τ and 2i -τ:

$$a\lambda\sum_i[A_{i,\tau}\zeta^{2i+\tau} + A_{i,-\tau}\zeta^{2i-\tau}].$$

The inequality $u_{\lambda,\tau}\zeta^{-1}$ corresponding to this pair of values of ±τ is sought in the same form as that in which we represented the right-hand side:

$$u_{\lambda,\tau}\zeta^{-1} = a\lambda\sum[\lambda_{i,\tau}\zeta^{2i+\tau} + \lambda_{i,-\tau}\zeta^{2i-\tau}].$$

We substitute this expression for $u_{\lambda,\tau}\zeta^{-1}$ and the conjugate quantity $s_{\lambda,\tau}\xi$ into the equation that derives from (9) by rejecting all groups of terms except the two chosen groups. Inasmuch as $\lambda_{i,\tau}$ and $\lambda_{i,-\tau}$ must satisfy this equation, they are determined from the condition that the coefficients on the right- and left-hand sides of the equation for identical powers of ζ be equal. This computed by Hill and Adams, which represent c_0 and g_0 as a function of the parameter m (refs. 10 and 19). After the values of c_0 and g_0 have been found, one of the unknown coefficients can be chosen arbitrarily, the rest being determined in the usual manner.

The coefficients of the zero-order inequalities

$$u_0\zeta^{-1} = a\sum a_i\zeta^{2i}, \qquad s_0\zeta = a\sum a_i\zeta^{2i}$$

will be regarded as known, since they have been obtained by Hill as power series in the parameter m (refs. 17 and 20).

3. FORMULATION OF THE PROGRAM AND MEMORY ALLOCATION /646

The process of computing the unknown coefficients of a complex of inequalities with a given characteristic can be broken down into a number of stages, including the following:

1) computation of the terms of the right-hand members of equations (7) or (8), generated by differentiation of the perturbation function;

2) computation of the terms of the right-hand members of equations (7) or (8), generated by series expansion of $u\zeta^{-1}/r^3$ or $z\sqrt{-1}/r^3$ (terms of equations (7) or (8) included in the brackets);

3) computation of the terms of the right-hand members of equations (7) or (8) generated by the expressions

$$-\zeta^{-1}(D^2+2mD)\sum u_\mu, \qquad D^2\sum z_\mu\sqrt{-1}$$

4) Determination of the unknown coefficients of the inequalities from equations (7) or (8).

During the operation, transition to the next stage is realized only after completion of the preceding computational stage. The sequence of operations embracing all four stages forms a cycle, which is repeated as many times as there are different complexes of inequalities to be computed in the total process.

The program of the entire operation is not stored in the internal memory of the computer. For this reason, it is written on magnetic tape and, as needed, the parts of the program used to carry out the individual computational stages are transferred into the operational (internal) memory.

The program of each of the indicated stages can be further broken down into a number of smaller sections, or program blocks. The blocks are named in accordance with the functions that they perform in the program.

The operational memory is divided into several banks.

Bank I (60 cells) stores the executive routine. One function of the master program is execution of the transition from one of the four computational stages to the next.

Bank II (130 cells) stores the characteristics of the complexes of inequalities at the beginning of operation.

Bank III (120 cells) stores the quantities m, c_0, g_0, M_i, N_i, P_i, Q_i,.... and the coefficients of the zero-order inequalities, a_i. All of these quantities are computed beforehand and are fed into memory at the beginning of operation.

Bank IV (500 cells) is set apart to store the coefficients of the first- to third order inequalities and the coefficient of the series expansions of c and g. This bank is filled up during the course of operation as the unknown coefficients are computed. The coefficients of some of the third-order inequalities and all higher order inequalities are written on magnetic tape.

The cells of bank V are used for working storage.

Bank VI consists of 365 cells. One section of these cells is used to record the right-hand members of the equations (7) and (8) as they are computed, another to record the results of intermediate multiplications, and a third to record the coefficients of the complexes of inequalities as they are transferred from tape.

Banks VII (430 cells) and VIII (260 cells) contain the routine for the next stage of computations. Specifically, in the first and second stages of bank VIII are allocated the executive block of the program and the coefficients of the expansions of the derivative of the perturbation function and $u\zeta^{-1}/r^3$ or $z\sqrt{-1}/r^3$.

The most complex routine governs the first stage of computations. It is written so that the transition to the second stage requires only replacement of the executive block of the program. The third-stage program is designed to make use of the individual blocks of the first-stage program. The fourth stage of computations requires a special program.

Logical diagrams are used below to record the individual blocks of the program. The following system of notation is used for the various types of operators:

Arithmetic operator A; transmit operator T; address substitution operator /647
F; reset operator O; forming operator Φ; logical operator P; unconditional transfer operator E; return-to-subroutine operator E_1.

The symbols are arranged in the logical diagrams in the order in which the operators function. If the operator transfers control to other than its next neighbor to the right, this is so indicated by an arrow. For each operator depending on the parameters, all of the parameters on which it depends is listed in brackets. For the reset, address substitution, and forming operators, it is indicated which operator performs the given operation. For the logical operators, the logical statement to be tested is normally indicated. In the event the logic statement is not satisfied, transfer of control is indicated by an arrow. If the transfer of control is executed to a block whose diagram is given in another figure, the end of the arrow is marked by the number of this block and the number of the operator to which control is to be transferred. The receiving of control from operators of other blocks is indicated analogously.

The program blocks are assigned the following numbers:

I) partitioning block; II) block for determination of initial cells; III) block for computation of the arguments; IV) block for determining addresses of the products; V) block for analysis of the type of multiplication; VI) multiplication block; VII) multiplication check block; VIII) block for multiplication by $K\zeta^{\beta} a_f^*$; IX) master program block for computation of terms in the right-hand member of equation (7), generated by $\partial\Omega/\partial s$; X) master program block for computation of terms in the right-hand member of equation (8), generated by $\frac{\partial\Omega}{\partial z}\sqrt{-1}$; XI) master program block for computation of terms in the right-hand member of equation (7), generated by series expansion of $u\zeta^{-1}/r^3$; XII) master program block for computation of terms in the right-hand member of equation (8), generated by series expansion of $z\sqrt{-1}/r^3$; XIII) executive routine for the total operation.

4. SELECTION OF TERMS WITH CHARACTERISTIC λ FROM THE EXPANSIONS OF $\frac{\partial\Omega}{\partial s}\zeta^{-1}$ AND $\frac{\partial\Omega}{\partial z}\sqrt{-1}$

Computing the partial derivatives of Ω with respect to s and z, the expressions for $-\frac{\partial\Omega}{\partial s}\zeta^{-1}$ and $-\frac{1}{2}\frac{\partial\Omega}{\partial z}\sqrt{-1}$ appearing in the right-hand members of equations (7) and (8) can be written in the following form:

$$
\begin{aligned}
-\frac{\partial\Omega}{\partial s}\zeta^{-1} = -am^2\Big\{&\left[\frac{3}{2}\cdot\frac{s\zeta}{\mathbf{a}}\cdot\zeta^{-2}\bar{a}_2+\frac{1}{2}\cdot\frac{u\zeta^{-1}}{\mathbf{a}}\cdot\zeta^0\cdot b_2\right]+\\
&+\frac{\mathbf{a}}{a'}\left[\frac{15}{8}\left(\frac{s\zeta}{\mathbf{a}}\right)^2\zeta^{-3}\bar{a}_3+\frac{3}{8}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)^2\zeta^1 c_3+\frac{3}{4}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)\left(\frac{s\zeta}{\mathbf{a}}\right)\zeta^{-1}\bar{c}_3+\frac{3}{2}\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)^2\bar{c}_3\zeta^{-1}\right]+\\
&+\frac{\mathbf{a}^2}{a'^2}\left[\frac{35}{16}\left(\frac{s\zeta}{\mathbf{a}}\right)^3\zeta^{-4}\bar{a}_4+\frac{5}{16}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)^3\zeta^2 c_4+\frac{15}{16}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)\left(\frac{s\zeta}{\mathbf{a}}\right)^2\zeta^{-2}\bar{c}_4+\frac{9}{16}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)^2\left(\frac{s\zeta}{\mathbf{a}}\right)\zeta^0 b_4+\right.\\
&\left.+\frac{15}{4}\left(\frac{s\zeta}{\mathbf{a}}\right)\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)^2\zeta^{-2}\bar{c}_4+\frac{9}{4}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)^2\zeta^0 b_4\right]\Big\},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
-\frac{1}{2}\frac{\partial\Omega}{\partial z}\sqrt{-1} = \mathbf{a}m^2\Big\{&\left[\frac{z\sqrt{-1}}{\mathbf{a}}\zeta^0 b_2\right]+\frac{\mathbf{a}}{a'}\left[\frac{3}{2}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)\zeta^{-1}c_3+\frac{3}{2}\left(\frac{s\zeta}{\mathbf{a}}\right)\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)\zeta^{-1}\bar{c}_3\right]+\\
&+\frac{\mathbf{a}^2}{a'^2}\left[\frac{15}{8}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)\zeta^2 c_4+\frac{15}{8}\left(\frac{s\zeta}{\mathbf{a}}\right)^2\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)\zeta^{-2}\bar{c}_4+\frac{9}{4}\left(\frac{u\zeta^{-1}}{\mathbf{a}}\right)\left(\frac{s\zeta}{\mathbf{a}}\right)\left(\frac{z\sqrt{-1}}{\mathbf{a}}\right)\zeta^0 b_4\right]\Big\}.
\end{aligned}
\tag{13}
$$

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The expressions contained in the brackets are called rows of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$.

Let us now consider the problem of how $\partial\Omega/\partial s$ and $\partial\Omega/\partial z$ can generate terms with the characteristic λ in the right-hand members of equations (7) and (8).

Everywhere in the expansions (12) and (13), $u\zeta^{-1}$, $s\zeta$, and $z\sqrt{-1}$ must be replaced by the sums $\sum u_\mu\zeta^{-1}$, $\sum s_\mu\zeta$, $\sum z_\mu\sqrt{-1}$, where μ runs through all possible characteristics of order less than the order of λ for a given coordinate (in the present case $u_0\zeta^{-1}$ and $s_0\zeta$ are included in $\Sigma u_\mu\zeta^{-1}$ and $\Sigma s_\mu\zeta$). Since u_μ, s_μ, and z_μ have the factor μ, with this substitution the products will contain terms with different characteristics. From amongst all the products it is required, in the case of the nth numbered row, to choose those which have the characteristic λ/a^{n-1}, since all the terms of this row have the factor $(A/a')^{n-1} = \alpha^{n-1}$ in front of the bracketed expression. The quantities $\bar{a}_2$, b_2, $\bar{a}_3$, ...appearing in the expressions for the individual terms of the expansions (12) and (13) are power series in the parameter e'. In these series, we separate expressions which comprise factors associated with e'^h and perform multiplication of the complexes of inequalities only by these expressions. Then, from all the possible products in each term of the expansion, it is necessary to select those which have the factor $\lambda/\alpha^{n-1}e'^{h}$.

Once all the products satisfying this condition have been chosen, we can go on to the new value of h and repeat the cycle of computations. After all values of h admissible for the characteristic λ have been exhausted, it is necessary to pass on to a new value of n and repeat the entire cycle of computations for the next row, beginning with the zero value of the parameter h. Consequently, two cycles of operations take place, one nested in the other, where the cycle in n is the outer one.

We turn now to the problem of selecting terms with the characteristic $\lambda/\alpha^{n-1}e'^{h}$ from all the possible products generated by an individual term of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$.

This problem can be formulated in a somewhat different fashion: for a given term of the expansion, find all admissible combinations of the n factor-characteristics having $\lambda/\alpha^{n-1}e'^{h}$ as their product. Each such combination will give the characteristics of those complexes of inequalities which must be multiplied in order to obtain a product with the characteristic $\lambda/\alpha^{n-1}e'^{h}$. In formulating the problem, particular mention is made of the fact that not every such breakdown of the characteristic into n factors can be utilized; the partitioning must be admissible for the given term. The need for pointing this out emerges from the fact that the complexes of inequalities of the coordinates u and s can have the characteristics containing the parameter $\mathbf{k}$ only in even powers, whereas $\mathbf{k}$ only enters in odd powers into the characteristics of complexes of inequalities of the coordinate z. Partitioning of the characteristic $\mathbf{k}^4$ into the factors $\mathbf{k}^2 \cdot \mathbf{k}^2$ is a permissible partition for the first, second, and third terms of the second row of the expansion of $\partial\Omega/\partial s$, while on the other hand the partitioning into $\mathbf{k}\cdot\mathbf{k}^3$ and $\mathbf{k}^3\cdot\mathbf{k}$ is inadmissible for these terms. In this connection is introduced the concept of "partition types." If the first of the split factors contains an even power of the parameter k, we say that the characteristic is subject to u-type partitioning. In the opposite case, we call it /649 z-type partitioning. If the characteristic is subject to partitioning into more than two factors, we speak of a sequence of partition types or of one complex partition. For example, in partitioning the characteristic $\lambda = \mathbf{k}e'^2\mathbf{e}^2$ into the factors $\mathbf{k}\cdot e'^2\cdot\mathbf{e}^2$, we are faced with a z-u-type partition; first the partition $\mathbf{k}\cdot e'^2\mathbf{e}^2$ (z-type), then partitioning of the second factor $e'^2\mathbf{e}^2$ into $e'^2\cdot\mathbf{e}^2$ (u-type). The sequence of admissible types of partitions for each term of the row can be specified by means of a logical scale. We agree to code the u-type partitions by 0, the z-type by 1. Then a complex partition of the type z-u has a code 10. The logical scale for the types of partitions in the third row of $\partial\Omega/\partial s$ is written out in the form

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0

1st term 2nd term 3rd term 4th term 5th term 6th term

It would be sufficient to allocate two places for each term of this row. The excess places are filled with zeros. In partitioning a certain characteristic λ into two factors, the first place is separated by a scanner probe from the places of the logical scale set aside for the term of the row subject to partitioning. The parity of the power of $\mathbf{k}$ in the first split factor is compared with the permitted parity determined from the logical scale. If the second factor is subject to further breakdown, the probe moves one place to the right and picks out the type of partition admissible for the second factor, and so on. Each partition of λ into several factors will be suitable for all subsequent terms of a single row for which the type of partition, coded in the next four sequential places, remains unchanged. This follows from the fact that both $u_\mu \zeta^{-1}$ and $s\zeta$ have the factor μ. Consequently, for example, if $u_\mu \zeta^{-1} u_\nu \zeta^{-1}$ has the factor λ, then so do $u_\mu \zeta^{-1} s_\nu \zeta$, $s_\mu \zeta\, u_\nu \zeta^{-1}$, $s_\mu \zeta s_\nu \zeta$. In correspondence with this, having obtained the next partition, we carry out all the necessary operations, for a given term of the row, on the series, then we go on to the next term (the logical scale shifting four places to the left relative to the position for the preceding term) and, in the event that the type of partition remains as before, we carry out, on the basis of the same partition, the operations needed for this term, etc. If in going on to the next term there is a change in the type of partition, we return along the row to the beginning of that group of terms of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ which has just been found in the operation (the logical scale must be reset in this case). The next stage in this case is to obtain a new partition of the same characteristic into the required number of factors. The process just described is repeated with the new partition. After all partitions of the characteristic admissible for a given group of terms have been exhausted, we move to the right along the row of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ to a new group of terms which have the same type of partition. Then the logical scale is shifted to the left so that the code for the admissible type of partition for the first term of the new group will be in the first four places of the memory cell.

The characteristic λ is again subjected to all admissible partitions for the new group, on the basis of which a certain number of terms of the right-hand members of equations (7) or (8) is computed each time. When all the groups of terms of the one row have thus been exhausted, we go on to a new value of the parameter h. Using the new value of the characteristic $\lambda_1 = \lambda/\alpha^{n-1} e'^{h+1}$, providing of course that it contains only negative powers of the parameter e', the whole computational cycle embracing the selection of terms with the characteristic λ_1 from the same row of the expansion must be repeated. Otherwise, we carry out address substitution of the cycles in n and reset according to the parameter h, and, if λ/α^n does not contain negative powers of the parameter α, we execute the computations for a new row. It is necessary to go on to the next, or second, stage of computations for detecting the negative powers (see section 3). /650

5. PARTITION OF THE CHARACTERISTIC INTO TWO OR MORE FACTORS

We note first of all that the characteristic to be partitioned is always an order lower than the characteristic of the complex of inequalities whose coefficients are being determined at any given stage. This is because the partition is applied to $\lambda/\alpha^{n-1}e'^{h}$, rather than to λ, where n-1 and h cannot be equal to zero at the same time, since the $\bar{a}_2$ and b_2 series are devoid of a free term. Consequently, assuming that a certain characteristic is partitioned into parts of order no higher than the order of the characteristic being partitioned, we always obtain the characteristics of the inequalities with known coefficients.

As already mentioned, the machine memory stores the characteristics of all the inequalities whose coefficients are to be determined. The factors into which the characteristic (denoted by λ) is partitioned are terms of this block of characteristics.

Let us see how the characteristic $\mathbf{ek}^2$ can be broken down into two parts. It is apparent that this can only be accomplished in six ways:

1) $1 \cdot \mathbf{ek}^2$,	4) $\mathbf{ek} \cdot \mathbf{k}$,
2) $\mathbf{e} \cdot \mathbf{k}^2$,	5) $\mathbf{k}^2 \cdot \mathbf{e}$,
3) $\mathbf{k} \cdot \mathbf{ek}$,	6) $\mathbf{ek}^2 \cdot 1$,

(the zero-order inequality has a characteristic 1).

It is readily seen how the partitioning process can be formalized.

For the first factor μ, all the terms are picked out in sequence from the block of characteristics whose order does not exceed the order of the characteristic being partitioned. Each time, the order number j_1 of μ in the block of characteristics is fixed. If it turns out that λ/μ contains negative powers of the parameters $\alpha, \mathbf{e}$, e', $\mathbf{k}$, then that partition is rejected as inadmissible. In the order described in the preceding section, a test is made to determine whether the parity of the power of $\mathbf{k}$ in the first factor corresponds to an admissible type of partition. If correspondence does not exist, we choose the next characteristic for μ and repeat the computational cycle.

So as not to overburden the memory with the factors into which a given characteristic is partitioned (their number can be rather large), it is better to wait until all the necessary operations have been carried out on the complexes of inequalities on the basis of the preceding partition before going ahead with the next one.

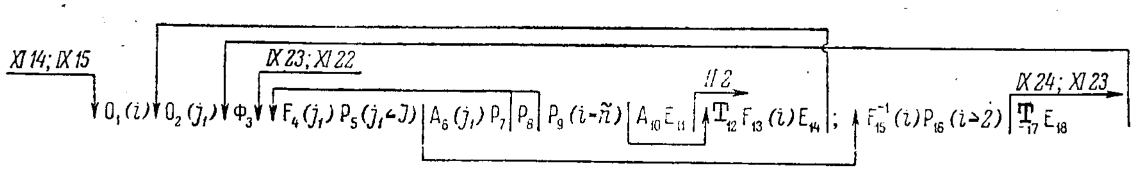
Let us now consider the case of partitioning into three or more factors. The second factor in each of the six alternatives for the partition of $\mathbf{ek}^2$ into two parts can be subjected in turn to partitioning into two factors, thus exhausting all possible partitions of $\mathbf{ek}$ into three parts. Consequently, the order of operations in the case of partition into three or more factors might be the following. Having obtained the next partition, for example 2) above, $\mathbf{e}\cdot\mathbf{k}^2$, we test to see whether the number of factors is sufficient for the given row of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$. If such is not the case, the second factor is subjected to all possible partitions into two parts. The procedure remains exactly as before. However, in order to be able to obtain the next partition 3) $\mathbf{k}\cdot\mathbf{ek}$ of the initial characteristic $\mathbf{ek}^2$ after exhausting all partitions of $\mathbf{k}^2$, in going to the stage of partitioning $\mathbf{k}^2$ we must store the characteristic partitioned in the preceding stage, as well as the order number j_1 of the latter characteristic, which has /651 already been tested in the preceding stage as the first factor. This can be accomplished by the following procedure.

The first of the factors into which the characteristic is partitioned is entered in the cell B_1, the second is entered in the cell B:

$$\begin{array}{ccccccccc} B_4 & B_3 & B_2 & B_1 & B & D_1 & D_2 & D_3 & D_4 \\ \langle j_4\rangle & \langle j_3\rangle & \langle j_2\rangle & \langle j_1\rangle & \langle j\rangle & & & & \end{array}$$

Then in the cell numbered $\langle j_1\rangle$ we enter j_1, the order number of the first factor in the block of characteristics. In the cell D_1 is stored the characteristic to be partitioned at this stage. If the number of factors into which the characteristic is partitioned proves to be insufficient, we transfer the contents of D_3 to D_4, D_2 to D_3, D_1 to D_2, and B to D_1. Figuratively speaking, the shift pulse moves to the right from D_3 to B. Similarly, we let a shift pulse to the left move from B_3 to B_1 and from $\langle j_3\rangle$ to $\langle j_1\rangle$. As a result, the necessary quantities will be fixed in the cells D_2, $\langle j_2\rangle$, B_2. The new contents of the cell D_1 is subjected to partitioning and the new pair of factors is entered into B_1 and B. If the number of factors is still not enough, the process is repeated. Finally, the cells B, B_1, B_2,... will contain the required number of factors, the cells $\langle j\rangle$, $\langle j_1\rangle$, $\langle j_2\rangle$,... will contain their order numbers in the block of characteristics (j, the order number of the factor contained in the cell B, is determined by comparing the contents of this cell with the cells of the characteristic block). The set of numbers j, j_1, j_2, j_3,... comprises that information relative to the partition, which is needed for operation of the subsequent program blocks. To obtain the second and subsequent partitions into n factors, the contents of D_1 are subjected to further partitioning into two factors. In this case, only the contents of the cells B_1, B, $<j_1>$, and $<j>$ will change. Each new partition is put into operation, and only then do we obtain the next partition. After all possible partitions of the contents of D_1 into two factors have been exhausted, it is required to obtain the next partition of the preceding stage. It is required primarily for this to renew the contents of the cells D_1, B_1, and $<j_1>$ at the instant of the last partition of the preceding stage. This can be done by transferring the contents of the cells D_2 into D_1, D_3 to D_2, D_4 to D_3 (shift pulse to the left) and by a shift pulse to the right, proceeding from B_2 to B_4 and from $<j_2>$ to $<j_4>$. Clearly, after the next partition of the preceding stage, the number of factors will be insufficient and it will be necessary to go on to the next higher stage, as already described. As a result, new quantities appear in the cells B_2, $<j_2>$, and D_1. The characteristic transferred into cell D_1 is again subjected to partitioning, and so on. It may happen that all partitions of the preceding stage have already been exhausted. In such event, we proceed just as in the exhausting of the partitions in the highest stage, i.e., we effect a transition to the next lower stage. The process of partitioning a given characteristic ends when all partitions of each stage have been exhausted.

An operational flowchart of the partition block may be written as follows:



The operator O_1 receives control from the executive block. The initial characteristic must be transferred to the cell D_1 in this case. It renews the initial value of the parameter i (i = 2) and position of the probe (see P_8).

O_2 renews the value of the parameter j_1 ($j_1 = 0$).

Φ_3, on the basis of the order of characteristic $\tilde{\lambda}$ to be partitioned at a given stage (found in the cell D_1), forms the largest value J that can be assumed by the parameter j_1 (J is equal to the number of characteristics whose order is no greater than the order of the characteristic $\tilde{\lambda}$).

P_5 tests fulfillment of the indicated logical statement. /652

A_6 selects from the block of characteristics the one with the number j_1 and transfers it to the cell B_1; it forms $\tilde{\lambda}/\mu$ and enters it in cell B.

P_7 tests fulfillment of the following condition: $\tilde{\lambda}/\mu$ contains no negative powers of the parameters $\mathbf{e}$, e', α, $\mathbf{k}$.

P_8, by means of the scanner probe segregates from the logic scale the admissible type of partition; it tests fulfillment of the following condition: The partition actually obtained coincides with the admissible type.

P_9 tests fulfillment of the following condition: $1 = \tilde{n}$, where $\tilde{n}$ is the number of factors into which the initial characteristic must be partitioned.

A_{10} determines the order number j of the characteristic located in the cell B.

E_{11} transfers control to the block for determining the initial cells.

T_{12} transmits the contents of the cells (shift pulse) $D_k \rightarrow D_{k+1}$, $B \rightarrow D_1$, $B_k \rightarrow B_{k+1}$, $\langle j_k \rangle \rightarrow \langle j_{k+1} \rangle$, beginning with the largest value of the parameter k.

F_{13} increments the parameter i once; it shifts the probe one place to the right.

F_{15}^{-1} decrements the parameter i once; it shifts the probe one place to the left.

P_{16} tests fulfillment of the following condition: $i \geq 2$. If the condition is not fulfilled, it transfers control to the executive routine, which executes transition to a new group of terms of the same row.

T_{17} transmits the contents of the cells $D_k \rightarrow D_{k-1}$, $B_k \rightarrow B_{k-1}$, $\langle j_k \rangle \rightarrow \langle j_{k-1} \rangle$, beginning with the smallest value of the parameter k.

6. STORING THE COEFFICIENTS OF THE INEQUALITIES IN MACHINE MEMORY

Below we will write the inequalities in a form differing somewhat from that used in section 2. Inasmuch as the arguments $\tau_1, \tau_2, \ldots, \tau_k$ are computed during the course of operation according to equation (5), which gives both the values of $+\tau$ and $-\tau$, it is sensible to write the complex of inequalities with characteristic λ in the coordinate u as follows:

$$\frac{u_\lambda \zeta^{-}}{\mathbf{a}} = \lambda \sum_{i,\tau} \lambda_{i,\tau} \zeta^{2i+\tau} \qquad \lambda = \mathbf{e}^{i_1} e'^{i_2} \mathbf{k}^{i_3} \alpha^{i_4}, \tag{14}$$

assuming that τ runs through all values defined by equation (5). We will consider that the groups of the complex which have the form

$$\sum_i \lambda_{i,\tau} \zeta^{2i+\tau},$$

are numbered in the order in which the arguments of the groups τ_k are computed during the course of operation according to equation (5).

It is supposed in equation (14) that for i_4 odd, i assumes values of $\pm 1/2$, $\pm 3/2, \ldots$ In this case, we will speak of a series in odd powers of ζ, otherwise of a series in even powers.

For programming purposes, it is convenient to assume that the index of the coefficient in every case assumes integer values. Consequently, in the case of series in odd powers of ζ we will write equation (14) in the form

$$\frac{u_\lambda \zeta^{-1}}{\mathbf{a}} = \lambda \sum_{i,\tau} \lambda_{i,\tau} \zeta^{2i-1+\tau}, \tag{14a}$$

assuming that i obtains integer values in this case as well.

A very important problem is to establish reasonable limits of variation of the index i. The choice of these limits rests, on the one hand, on the desired accuracy of the computations, on the other hand on the capacity of the opera- /653 tional memory. Expanding the limits of variation of the index i, of course, increases the machine time for the task.

The quantities $\lambda_{i,\tau}$, in general, decrease rapidly with growing $|i|$. For instance, the coefficients a_i of the zero-order inequality have a value of the order $m^{|2i|}$. For higher order inequalities, the coefficients $\lambda_{i,\tau}$ can have orders with respect to m several units higher or lower than $|2i|$ (the order always remains nonnegative), these deviations having a tendency to increase with increasing orders of the inequalities. If the coefficients of the inequalities of different order were to be computed with the same accuracy, the situation just noted would necessitate an increase in the limits of variation of the index i with increasing orders. This is not the case in reality. The reason for the "scatter" of orders of the coefficients is found in the small divisors which arise in determining certain coefficients from the systems of equations (10) and (11). A small divisor not only lowers the order of the coefficient, it also lowers the accuracy with which it is determined. In view of this, the right-hand members of the equations for determining the coefficients of higher order inequalities are computed, generally speaking, with greater error than the right-hand members of the equations for determining the coefficients of the inequalities of the preceding orders. Consequently, there occurs a progressive loss of accuracy with which the coefficients of the various orders of inequalities can be computed. On the other hand, it is all right to know the higher order inequalities with lower accuracy, since they are multiplied by smaller characteristics. Thus, in part at least, the loss of accuracy in determining the coefficients with increasing orders of the inequalities is compensated. In view of the fact that the "scatter" of the orders proceeds in parallel with the reduction in computational accuracy, the limits of variation of i for which all nonnegligible terms are taken into account are not taken into account with increasing orders of the inequalities. On the contrary, they are brought closer together, albeit slowly. In formulating his lunar theory, Brown was not compelled in any case to use a value of $|i|$ any higher than five, and for inequalities in odd powers of ζ, never a value of $|2i|$ higher than ten. For the satellites of Jupiter (VI, VII, and X), requiring lower accuracy, the limits can be fixed even narrower. This is confirmed by the work of S. S. Tokmalayeva in determining the perturbations of the satellite VII.

We will assume below that i varies within the limits $-4 \leqslant i \leqslant +4$, and in those cases when the series is in odd powers of ζ, $-3 \leqslant i \leqslant +4$. Ten cells of memory are allocated for the coefficients of each group of terms, these cells containing in order of increasing cell address

$$\lambda_{-4,\tau},\ \lambda_{-3,\tau},\ \lambda_{-2,\tau},\ \lambda_{-1,\tau},\ \lambda_{0,\tau},\ \lambda_{1,\tau},\ \lambda_{2,\tau},\ \lambda_{3,\tau},\ \lambda_{4,\tau};$$

which are the coefficients associated respectively with

$$\zeta^{-8+\tau},\ \zeta^{-6+\tau},\ \zeta^{-4+\tau},\ \zeta^{-2+\tau},\ \zeta^{\tau},\ \zeta^{2+\tau},\ \zeta^{4+\tau},\ \zeta^{6+\tau},\ \zeta^{8+\tau}.$$

In the case of series in odd powers of ζ, we agree to leave the first cell of the group empty, placing in the remaining cells

$$\lambda_{-3,\tau},\ \lambda_{-2,\tau},\ \lambda_{-1,\tau},\ \lambda_{0,\tau},\ \lambda_{1,\tau},\ \lambda_{2,\tau},\ \lambda_{3,\tau},\ \lambda_{4,\tau},$$

which are the coefficients associated respectively with

$$\zeta^{-7+\tau},\ \zeta^{-5+\tau},\ \zeta^{-3+\tau},\ \zeta^{-1+\tau},\ \zeta^{1+\tau},\ \zeta^{3+\tau},\ \zeta^{5+\tau},\ \zeta^{7+\tau}.$$

A cell corresponding to the index i = 0 will be called the <u>initial cell of the group</u> of coefficients, and whenever the group is the first one in the complex, the cell in which λ_{0,τ_1} is stored will be called the <u>initial cell of the complex</u> of inequalities. Knowing the group number k and the number N_0 of the initial cell of the complex, it is a simple matter to calculate the number of the initial cell of the group. It is plainly equal to

$$N_0 + 9(k-1). \tag{15}$$

The number of the cell of this group in which the coefficient λ_{i,τ_k} is stored is equal to /654

$$N_0 + 9(k-1) + i.$$

Inasmuch as the coordinate s is the complex conjugate of the coordinate u, the coefficients of the inequalities of this coordinate can be easily derived if the coefficients of the inequalities of the coordinate u are known. Substituting ζ and ζ^{-1} into equations (14) and (14a), we obtain, respectively,

$$\frac{s_\lambda \zeta}{\mathbf{a}} = \lambda \sum_{i,\tau} \lambda_{i,\tau} \zeta^{-2i \text{[illegible]}} = \lambda \sum_{i,\tau} \lambda_{i,-\tau} \zeta^{-2i \text{[illegible]}}, \tag{16}$$

$$\frac{s_\lambda \zeta}{\mathbf{a}} = \lambda \sum_{i,\tau} \lambda_{i,\tau} \zeta^{-2i+ \text{[illegible]}} = \lambda \sum_{i,\tau} \lambda_{i,-\tau} \zeta^{-2i+ \text{[illegible]}}. \tag{16a}$$

Equations (16) and (16a) show that the initial cell of the group with argument $+\tau_k$ in the complex $s_\lambda \zeta / \mathbf{a}$ is the initial cell of the group with argument $-\tau_k$ in the complex $u_\lambda \zeta / \mathbf{a}$. It is not too difficult to compute its number from equation (15), for the argument $-\tau_k$ has the order number K - (k - 1), where K is the total number of arguments of the inequality, and k is the order number of the argument $+\tau_k$ (see section 9). Thus, the number of the initial cell in this case is equal to

$$N_{\mu} + 9(K-1) - 9(k-1). \tag{17}$$

For the inequalities of the coordinate z, we adopt the following form of writing:

$$\frac{z_\mu \sqrt{-1}}{a} = \ldots \sum \mu_{i,\tau}\zeta^{\ldots} + \sum_{i,\tau} \mu_{-i,-\tau}\zeta^{-2i\ldots} \Big],$$

assuming that underneath the first summation sign τ takes on the first half of the values determined by equation (5) (it is readily seen that for $z_\mu\sqrt{-1}$ the number of different τ is always even), while underneath the second summation sign τ takes on the second half of the values determined by equation (5). If we make use of equation (6), we can write

$$\frac{\ldots}{a} = \mu \Big[\sum_{i,\tau} \mu_{i,\tau}\zeta^{\ldots} + \sum_{i,\tau} \ldots \mu_{i,\ldots}\zeta^{\ldots+\tau} \Big] \tag{18}$$

and, if α enters into the characteristic in an odd power,

$$\frac{z_\mu\sqrt{-1}}{a} = \mu \Big[\sum_{i,\tau} \mu_{i,\tau}\zeta^{2i-1+\tau} + \sum_{i,\tau} -\mu_{i,-\tau}\zeta^{-2i+1+\tau} \Big] \tag{18a}$$

Comparison of these two expressions with equations (14), (14a) and (16), (16a) shows that in the case of the coordinate z we must proceed in order to compute the initial cell of the group with argument τ_k, when $k \leqslant K/2$, the same as for the coordinate u, i.e., calculate the number of the initial cell from equation (15); for $k > K/2$, it is necessary to use equation (17) for the coordinate s. In this case, of course, it is essential to bear in mind that the sign of the coefficients involved must be changed in the second case.

Because in the derivation of equations (18) and (18a), we made use of the relation $\tau_{-i,\tau_k} = -\mu_{i,-\tau_k}$, which exists between the coefficients of the groups with argument $+\tau_k$ and $-\tau_k$, it is sufficient to store in memory only the coefficients of the first half of all the groups of the complex of inequalities of the coordinate z. Applying the rules cited above, we will always invoke the initial cells of the first half of all the groups.

7. TYPES OF SERIES MULTIPLICATION /655

In multiplying the series by each other, we will distinguish four types of multiplication, depending on whether the terms in the cross-multiplied series are arranged in descending or ascending powers of ζ.

Type 1 multiplication. - Both series are arranged in ascending powers of ζ. This type of multiplication occurs in the multiplication of groups of terms in the coordinate u:

$$\sum_i \lambda_{i,\tau}\zeta^{2i+\tau} \cdot \sum_i \mu_{i,\sigma}\zeta^{2i+\sigma} = \sum_i \sum_k \lambda_{i-k,\tau}\mu_{k,\sigma}\zeta^{2i+\tau+\sigma}.$$

Type 2 multiplication. - The first series is arranged in ascending powers of ζ, the second in descending powers, which occurs, for example, when a group of terms in the coordinate u is multiplied by a group of terms in the coordinate s:

$$\sum_i \lambda_{i,\tau}\zeta^{2i+\tau} \cdot \sum_i \mu_{i,\sigma}\zeta^{-2i+\sigma} = \sum_i \sum_k \lambda_{i-k,\tau}\mu_{-k,\sigma}\zeta^{2i+\tau+\sigma}.$$

Type 3 multiplication. - The first term is arranged in descending powers of ζ, the second in ascending powers. This occurs when a group of terms in the coordinate s is multiplied by a group of terms in the coordinate u:

$$\sum_i \lambda_{i,\tau}\zeta^{-2i+\tau} \cdot \sum_i \mu_{i,\sigma}\zeta^{2i+\sigma} = \sum_i \sum_k \lambda_{i-k,\tau}\mu_{-k,\sigma}\zeta^{-2i+\tau+\sigma}$$

Type 4 multiplication. - Both series are arranged in descending powers of ζ. This occurs in the multiplication of groups in the coordinate s:

$$\sum_i \lambda_{i,\tau}\zeta^{-2i+\tau} \cdot \sum_i \mu_{i,\sigma}\zeta^{-2i+\sigma} = \sum_i \sum_k \lambda_{i-k,\tau}\mu_{k,\sigma}\zeta^{-2i+\tau+\sigma}.$$

For the sake of simplicity in writing, we will consistently represent series expanded in even powers of ζ. We note one fact that will be used later on: The coefficients of $\zeta^{-2i+\tau+\sigma}$ in the product series in the third and fourth types are equal to the coefficients $\zeta^{2i+\tau+\sigma}$ in the product series in the second and first types, respectively.

If the series is expanded in even powers of ζ, we say that it has a parity index equal to zero, otherwise it has a parity index equal to one. The corresponding parity index will be represented by a superscript 0 or 1 alongside the series symbol. Then, in multiplying two groups of terms, we may encounter the following cases:

Type 1 multiplication	Type 2 multiplication
1) $u^0 \cdot u^0$,	1) $u^0 \cdot s^0$,
2) $u^1 \cdot u^0$,	2) $u^1 \cdot s^0$,
3) $u^0 \cdot u^1$,	3) $u^0 \cdot s^1$,
4) $u^1 \cdot u^1$,	4) $u^1 \cdot s^1$,

where u and s denote the groups of terms arranged in ascending or descending powers of ζ, respectively.

For greater ease of visualization, these products are presented below in tabular form. In the tables, the sum of the products in one column is equal to the product series coefficient d_i at the top of the column. Since the indices τ and σ are not significant for our purposes, they are not written out, and the coefficients $\lambda_{i,\tau}$ and $\mu_{i,\tau}$ are replaced by b_i, c_i.

Type 1 multiplication

1st case:

$$\begin{aligned}&(b_{-4}\zeta^{-8}+b_{-3}\zeta^{-6}+b_{-2}\zeta^{-4}+b_{-1}\zeta^{-2}+b_0\zeta^0+b_1\zeta^2+b_2\zeta^4+b_3\zeta^6+b_4\zeta^8)\times\\ &\times(c_{-4}\zeta^{-8}+c_{-3}\zeta^{-6}+c_{-2}\zeta^{-4}+c_{-1}\zeta^{-2}+c_0\zeta^0+c_1\zeta^2+c_2\zeta^4+c_3\zeta^6+c_4\zeta^8)=\\ &=d_{-4}\zeta^{-8}+d_{-3}\zeta^{-6}+d_{-2}\zeta^{-4}+d_{-1}\zeta^{-2}+d_0\zeta^0+d_1\zeta^2+d_2\zeta^4+d_3\zeta^6+d_4\zeta^8\end{aligned}$$

$$\begin{array}{ccccccccc}
b_0c_{-4} & b_1c_{-4} & b_2c_{-4} & b_3c_{-4} & b_4c_{-4} & & & & \\
b_{-1}c_{-3} & b_0c_{-3} & b_1c_{-3} & b_2c_{-3} & b_3c_{-3} & b_4c_{-3} & & & \\
b_{-2}c_{-2} & b_{-1}c_{-2} & b_0c_{-2} & b_1c_{-2} & b_2c_{-2} & b_3c_{-2} & b_4c_{-2} & & \\
b_{-3}c_{-1} & b_{-2}c_{-1} & b_{-1}c_{-1} & b_0c_{-1} & b_1c_{-1} & b_2c_{-1} & b_3c_{-1} & b_4c_{-1} & \\
b_{-4}c_0 & b_{-3}c_0 & b_{-2}c_0 & b_{-1}c_0 & b_0c_0 & b_1c_0 & b_2c_0 & b_3c_0 & b_4c_0 \\
 & b_{-4}c_1 & b_{-3}c_1 & b_{-2}c_1 & b_{-1}c_1 & b_0c_1 & b_1c_1 & b_2c_1 & b_3c_1 \\
 & & b_{-4}c_2 & b_{-3}c_2 & b_{-2}c_2 & b_{-1}c_2 & b_0c_2 & b_1c_2 & b_2c_2 \\
 & & & b_{-4}c_3 & b_{-3}c_3 & b_{-2}c_3 & b_{-1}c_3 & b_0c_3 & b_1c_3 \\
 & & & & b_{-4}c_4 & b_{-3}c_4 & b_{-2}c_4 & b_{-1}c_4 & b_0c_4
\end{array}$$

2nd case:

$$\begin{aligned}&(b_{-3}\zeta^{-7}+b_{-2}\zeta^{-5}+b_{-1}\zeta^{-3}+b_0\zeta^{-1}+b_1\zeta^1+b_2\zeta^3+b_3\zeta^5+b_4\zeta^7)\times\\ &\times(c_{-4}\zeta^{-8}+c_{-3}\zeta^{-6}+c_{-2}\zeta^{-4}+c_{-1}\zeta^{-2}+c_0\zeta^0+c_1\zeta^2+c_2\zeta^4+c_3\zeta^6+c_4\zeta^8)=\\ &=d_{-3}\zeta^{-7}+d_{-2}\zeta^{-5}+d_{-1}\zeta^{-3}+d_0\zeta^{-1}+d_1\zeta^1+d_2\zeta^3+d_3\zeta^5+d_4\zeta^7\end{aligned}$$

$$\begin{array}{cccccccc}
b_1c_{-4} & b_2c_{-4} & b_3c_{-4} & b_4c_{-4} & & & & \\
b_0c_{-3} & b_1c_{-3} & b_2c_{-3} & b_3c_{-3} & b_4c_{-3} & & & \\
b_{-1}c_{-2} & b_0c_{-2} & b_1c_{-2} & b_2c_{-2} & b_3c_{-2} & b_4c_{-2} & & \\
b_{-2}c_{-1} & b_{-1}c_{-1} & b_0c_{-1} & b_1c_{-1} & b_2c_{-1} & b_3c_{-1} & b_4c_{-1} & \\
b_{-3}c_0 & b_{-2}c_0 & b_{-1}c_0 & b_0c_0 & b_1c_0 & b_2c_0 & b_3c_0 & b_4c_0 \\
 & b_{-3}c_1 & b_{-2}c_1 & b_{-1}c_1 & b_0c_1 & b_1c_1 & b_2c_1 & b_3c_1 \\
 & & b_{-3}c_2 & b_{-2}c_2 & b_{-1}c_2 & b_0c_2 & b_1c_2 & b_2c_2 \\
 & & & b_{-3}c_3 & b_{-2}c_3 & b_{-1}c_3 & b_0c_3 & b_1c_3 \\
 & & & & b_{-3}c_4 & b_{-2}c_4 & b_{-1}c_4 & b_0c_4
\end{array}$$

3rd case:

$$\begin{aligned}&(b_{-4}\zeta^{-8}+b_{-3}\zeta^{-6}+b_{-2}\zeta^{-4}+b_{-1}\zeta^{-2}+b_0\zeta^0+b_1\zeta^2+b_2\zeta^4+b_3\zeta^6+b_4\zeta^8)\times\\ &\times(c_{-3}\zeta^{-7}+c_{-2}\zeta^{-5}+c_{-1}\zeta^{-3}+c_0\zeta^{-1}+c_1\zeta^1+c_2\zeta^3+c_3\zeta^5+c_4\zeta^7)=\\ &=d_{-3}\zeta^{-7}+d_{-2}\zeta^{-5}+d_{-1}\zeta^{-3}+d_0\zeta^{-1}+d_1\zeta^1+d_2\zeta^3+d_3\zeta^5+d_4\zeta^7\end{aligned}$$

$$\begin{array}{cccccccc}
b_0c_{-3} & b_1c_{-3} & b_2c_{-3} & b_3c_{-3} & b_4c_{-3} & & & \\
b_{-1}c_{-2} & b_0c_{-2} & b_1c_{-2} & b_2c_{-2} & b_3c_{-2} & b_4c_{-2} & & \\
b_{-2}c_{-1} & b_{-1}c_{-1} & b_0c_{-1} & b_1c_{-1} & b_2c_{-1} & b_3c_{-1} & b_4c_{-1} & \\
b_{-3}c_0 & b_{-2}c_0 & b_{-1}c_0 & b_0c_0 & b_1c_0 & b_2c_0 & b_3c_0 & b_4c_0 \\
b_{-4}c_1 & b_{-3}c_1 & b_{-2}c_1 & b_{-1}c_1 & b_0c_1 & b_1c_1 & b_2c_1 & b_3c_1 \\
 & b_{-4}c_2 & b_{-3}c_2 & b_{-2}c_2 & b_{-1}c_2 & b_0c_2 & b_1c_2 & b_2c_2 \\
 & & b_{-4}c_3 & b_{-3}c_3 & b_{-2}c_3 & b_{-1}c_3 & b_0c_3 & b_1c_3 \\
 & & & b_{-4}c_4 & b_{-3}c_4 & b_{-2}c_4 & b_{-1}c_4 & b_0c_4
\end{array}$$

4th case:

$$(b_{-3}\zeta^{-7}+b_{-2}\zeta^{-5}+b_{-1}\zeta^{-3}+b_0\zeta^{-1}+b_1\zeta^1+b_2\zeta^3+b_3\zeta^5+b_4\zeta^7)\times$$
$$\times(c_{-3}\zeta^{-7}+c_{-2}\zeta^{-5}+c_{-1}\zeta^{-3}+c_0\zeta^{-1}+c_1\zeta^1+c_2\zeta^3+c_3\zeta^5+c_4\zeta^7)=$$
$$=d_{-4}\zeta^{-8}+d_{-3}\zeta^{-6}+d_{-2}\zeta^{-4}+d_{-1}\zeta^{-2}+d_0\zeta^0+d_1\zeta^2+d_2\zeta^4+d_3\zeta^6+d_4\zeta^8$$

$\underline{b_0c_{-3}}$	$\underline{b_1c_{-3}}$	b_2c_{-3}	b_3c_{-3}	b_4c_{-3}				
$\underline{b_{-1}c_{-2}}$	$\underline{b_0c_{-2}}$	$\underline{b_1c_{-2}}$	$\underline{b_2c_{-2}}$	b_3c_{-2}	b_4c_{-2}			
$\underline{b_{-2}c_{-1}}$	$\underline{b_{-1}c_{-1}}$	$\underline{b_0c_{-1}}$	$\underline{b_1c_{-1}}$	$\underline{b_2c_{-1}}$	$\underline{b_3c_{-1}}$	b_4c_{-1}		
$\underline{b_{-3}c_0}$	$\underline{b_{-2}c_0}$	$\underline{b_{-1}c_0}$	b_0c_0	$\underline{b_1c_0}$	$\underline{b_2c_0}$	$\underline{b_3c_0}$	$\underline{b_4c_0}$	
	$\underline{b_{-3}c_1}$	$\underline{b_{-2}c_1}$	$\underline{b_{-1}c_1}$	$\underline{b_0c_1}$	$\underline{b_1c_1}$	$\underline{b_2c_1}$	$\underline{b_3c_1}$	$\underline{b_4c_1}$
		$b_{-3}c_2$	$\underline{b_{-2}c_2}$	$\underline{b_{-1}c_2}$	$\underline{b_0c_2}$	$\underline{b_1c_2}$	$\underline{b_2c_2}$	$\underline{b_3c_2}$
			$b_{-3}c_3$	$b_{-2}c_3$	$\underline{b_{-1}c_3}$	$\underline{b_0c_3}$	$\underline{b_1c_3}$	$\underline{b_2c_3}$
				$b_{-3}c_4$	$b_{-2}c_4$	$b_{-1}c_4$	$\underline{b_0c_4}$	$\underline{b_1c_4}$

Type 2 multiplication

1st case:

$$(b_{-4}\zeta^{-8}+b_{-3}\zeta^{-6}+b_{-2}\zeta^{-4}+b_{-1}\zeta^{-2}+b_0\zeta^0+b_1\zeta^2+b_2\zeta^4+b_3\zeta^6+b_4\zeta^8)\times$$
$$\times(c_{-4}\zeta^8+c_{-3}\zeta^6+c_{-2}\zeta^4+c_{-1}\zeta^2+c_0\zeta^0+c_1\zeta^{-2}+c_2\zeta^{-4}+c_3\zeta^{-6}+c_4\zeta^{-8})=$$
$$=d_{-4}\zeta^{-8}+d_{-3}\zeta^{-6}+d_{-2}\zeta^{-4}+d_{-1}\zeta^{-2}+d_0\zeta^0+d_1\zeta^2+d_2\zeta^4+d_3\zeta^6+d_4\zeta^8$$

$\underline{b_0c_4}$	b_1c_4	b_2c_4	b_3c_4	b_4c_4				
$\underline{b_{-1}c_3}$	$\underline{b_0c_3}$	b_1c_3	b_2c_3	b_3c_3	b_4c_3			
$\underline{b_{-2}c_2}$	$\underline{b_{-1}c_2}$	$\underline{b_0c_2}$	$\underline{b_1c_2}$	b_2c_2	b_3c_2	b_4c_2		
$\underline{b_{-3}c_1}$	$\underline{b_{-2}c_1}$	$\underline{b_{-1}c_1}$	$\underline{b_0c_1}$	$\underline{b_1c_1}$	$\underline{b_2c_1}$	$\underline{b_3c_1}$	b_4c_1	
$\underline{b_{-4}c_0}$	$\underline{b_{-3}c_0}$	$\underline{b_{-2}c_0}$	$\underline{b_{-1}c_0}$	$\underline{b_0c_0}$	$\underline{b_1c_0}$	$\underline{b_2c_0}$	$\underline{b_3c_0}$	$\underline{b_4c_0}$
	$b_{-4}c_{-1}$	$\underline{b_{-3}c_{-1}}$	$\underline{b_{-2}c_{-1}}$	$\underline{b_{-1}c_{-1}}$	$\underline{b_0c_{-1}}$	$\underline{b_1c_{-1}}$	$\underline{b_2c_{-1}}$	$\underline{b_3c_{-1}}$
		$b_{-4}c_{-2}$	$b_{-3}c_{-2}$	$\underline{b_{-2}c_{-2}}$	$\underline{b_{-1}c_{-2}}$	$\underline{b_0c_{-2}}$	$\underline{b_1c_{-2}}$	$\underline{b_2c_{-2}}$
			$b_{-4}c_{-3}$	$b_{-3}c_{-3}$	$b_{-2}c_{-3}$	$\underline{b_{-1}c_{-3}}$	$\underline{b_0c_{-3}}$	$\underline{b_1c_{-3}}$
				$b_{-4}c_{-4}$	$b_{-3}c_{-4}$	$b_{-2}c_{-4}$	$b_{-1}c_{-4}$	$\underline{b_0c_{-4}}$

2nd case:

$$(b_{-3}\zeta^{-7}+b_{-2}\zeta^{-5}+b_{-1}\zeta^{-3}+b_0\zeta^{-1}+b_1\zeta^1+b_2\zeta^3+b_3\zeta^5+b_4\zeta^7)\times$$
$$\times(c_{-4}\zeta^8+c_{-3}\zeta^6+c_{-2}\zeta^4+c_{-1}\zeta^2+c_0\zeta^0+c_1\zeta^{-2}+c_2\zeta^{-4}+c_3\zeta^{-6}+c_4\zeta^{-8})=$$
$$=d_{-3}\zeta^{-7}+d_{-2}\zeta^{-5}+d_{-1}\zeta^{-3}+d_0\zeta^{-1}+d_1\zeta^1+d_2\zeta^3+d_3\zeta^5+d_4\zeta^7$$

b_1c_4	b_2c_4	b_3c_4	b_4c_4				
$\underline{b_0c_3}$	$\underline{b_1c_3}$	b_2c_3	b_3c_3	b_4c_3			
$\underline{b_{-1}c_2}$	$\underline{b_0c_2}$	$\underline{b_1c_2}$	$\underline{b_2c_2}$	b_3c_2	b_4c_2		
$\underline{b_{-2}c_1}$	$\underline{b_{-1}c_1}$	$\underline{b_0c_1}$	$\underline{b_1c_1}$	$\underline{b_2c_1}$	$\underline{b_3c_1}$	b_4c_1	
$\underline{b_{-3}c_0}$	$\underline{b_{-2}c_0}$	$\underline{b_{-1}c_0}$	$\underline{b_0c_0}$	$\underline{b_1c_0}$	$\underline{b_2c_0}$	$\underline{b_3c_0}$	$\underline{b_4c_0}$
	$\underline{b_{-3}c_{-1}}$	$\underline{b_{-2}c_{-1}}$	$\underline{b_{-1}c_{-1}}$	$\underline{b_0c_{-1}}$	$\underline{b_1c_{-1}}$	$\underline{b_2c_{-1}}$	$\underline{b_3c_{-1}}$
		$b_{-3}c_{-2}$	$\underline{b_{-2}c_{-2}}$	$\underline{b_{-1}c_{-2}}$	$\underline{b_0c_{-2}}$	$\underline{b_1c_{-2}}$	$\underline{b_2c_{-2}}$
			$b_{-3}c_{-3}$	$b_{-2}c_{-3}$	$b_{-1}c_{-3}$	$\underline{b_0c_{-3}}$	$\underline{b_1c_{-3}}$
				$b_{-3}c_{-4}$	$b_{-2}c_{-4}$	$b_{-1}c_{-4}$	$\underline{b_0c_{-4}}$

3rd case:

$$(b_{-4}\zeta^{-8} + b_{-3}\zeta^{-6} + b_{-2}\zeta^{-4} + b_{-1}\zeta^{-2} + b_0\zeta^0 + b_1\zeta^2 + b_2\zeta^4 + b_3c^6 + b_4\zeta^8) \times$$
$$\times (c_{-3}\zeta^7 + c_{-2}\zeta^5 + c_{-1}\zeta^3 + c_0\zeta^1 + c_1\zeta^{-1} + c_2\zeta^{-3} + c_3\zeta^{-5} + c_4\zeta^{-7}) =$$
$$= d_{-3}\zeta^{-7} + d_{-2}\zeta^{-5} + d_{-1}\zeta^{-3} + d_0\zeta^{-1} + d_1\zeta^1 + d_2\zeta^3 + d_3\zeta^5 + d_4\zeta^7$$

$$\begin{array}{cccccccc}
\underline{b_0c_4} & \underline{b_1c_4} & b_2c_4 & b_3c_4 & b_4c_4 & & & \\
\underline{b_{-1}c_3} & \underline{b_0c_3} & \underline{b_1c_3} & \underline{b_2c_3} & b_3c_3 & b_4c_3 & & \\
\underline{b_{-2}c_2} & \underline{b_{-1}c_2} & \underline{b_0c_2} & \underline{b_1c_2} & \underline{b_2c_2} & \underline{b_3c_2} & b_4c_2 & \\
\underline{b_{-3}c_1} & \underline{b_{-2}c_1} & \underline{b_{-1}c_1} & \underline{b_0c_1} & \underline{b_1c_1} & \underline{b_2c_1} & \underline{b_3c_1} & \underline{b_4c_1} \\
b_{-4}c_0 & \underline{b_{-3}c_0} & \underline{b_{-2}c_0} & \underline{b_{-1}c_0} & \underline{b_0c_0} & \underline{b_1c_0} & \underline{b_2c_0} & \underline{b_3c_0} \\
 & b_{-4}c_{-1} & b_{-3}c_{-1} & \underline{b_{-2}c_{-1}} & \underline{b_{-1}c_{-1}} & \underline{b_0c_{-1}} & \underline{b_1c_{-1}} & \underline{b_2c_{-1}} \\
 & & b_{-4}c_{-2} & \underline{b_{-3}c_{-2}} & \underline{b_{-2}c_{-2}} & \underline{b_{-1}c_{-2}} & \underline{b_0c_{-2}} & \underline{b_1c_{-2}} \\
 & & & b_{-4}c_{-3} & b_{-3}c_{-3} & b_{-2}c_{-3} & b_{-1}c_{-3} & \underline{b_0c_{-3}}
\end{array}$$

4th case:

$$(b_{-3}\zeta^{-7} + b_{-2}\zeta^{-5} + b_{-1}\zeta^{-3} + b_0\zeta^{-1} + b_1\zeta^1 + b_2\zeta^3 + b_3\zeta^5 + b_4\zeta^7) \times$$
$$\times (c_{-3}\zeta^7 + c_{-2}\zeta^5 + c_{-1}\zeta^3 + c_0\zeta^1 + c_1\zeta^{-1} + c_2\zeta^{-3} + c_3\zeta^{-5} + c_4\zeta^{-7}) =$$
$$= d_{-4}\zeta^{-8} + d_{-3}\zeta^{-6} + d_{-2}\zeta^{-4} + d_{-1}\zeta^{-2} + d_0\zeta^0 + d_1\zeta^2 + d_2\zeta^4 + d_3\zeta^6 + d_4\zeta^8$$

$$\begin{array}{ccccccccc}
\underline{b_0c_4} & \underline{b_1c_4} & b_2c_4 & b_3c_4 & b_4c_4 & & & & \\
\underline{b_{-1}c_3} & \underline{b_0c_3} & \underline{b_1c_3} & \underline{b_2c_3} & b_3c_3 & b_4c_3 & & & \\
\underline{b_{-2}c_2} & \underline{b_{-1}c_2} & \underline{b_0c_2} & \underline{b_1c_2} & \underline{b_2c_2} & \underline{b_3c_2} & b_4c_2 & & \\
\underline{b_{-3}c_1} & \underline{b_{-2}c_1} & \underline{b_{-1}c_1} & \underline{b_0c_1} & \underline{b_1c_1} & \underline{b_2c_1} & \underline{b_3c_1} & \underline{b_4c_1} & \\
 & \underline{b_{-3}c_0} & \underline{b_{-2}c_0} & \underline{b_{-1}c_0} & \underline{b_0c_0} & \underline{b_1c_0} & \underline{b_2c_0} & \underline{b_3c_0} & \underline{b_4c_0} \\
 & & b_{-3}c_{-1} & \underline{b_{-2}c_{-1}} & \underline{b_{-1}c_{-1}} & \underline{b_0c_{-1}} & \underline{b_1c_{-1}} & \underline{b_2c_{-1}} & \underline{b_3c_{-1}} \\
 & & & b_{-3}c_{-2} & b_{-2}c_{-2} & \underline{b_{-1}c_{-2}} & \underline{b_0c_{-2}} & \underline{b_1c_{-2}} & \underline{b_2c_{-2}} \\
 & & & & b_{-3}c_{-3} & b_{-2}c_{-3} & b_{-1}c_{-3} & \underline{b_0c_{-3}} & \underline{b_1c_{-3}}
\end{array}$$

8. PRINCIPLE OF CONSTRUCTION OF THE GROUP MULTIPLICATION PROGRAM

The computation of the products cited in the preceding section can be approached differently. The objective may be to compute the coefficients of the products as the sums of all products standing in one column. The program to carry out this objective can be based on specification of the value of the indices i and k for the factors standing in the first row of the first column, followed by address substitution in i and in k and a test of the conditions $|i - k| \leqslant 4$, $|k| \leqslant 4$. However, such a program is uneconomical from the viewpoint of machine time, because the computation of each product is preceded by a large number of auxiliary operations, where a sizable portion of the address substitution has to be carried out as a dummy operation, no reading being made until the logical statements have been fulfilled. Finally, it must be pointed out that not all of the products in the columns need actually be computed, since many of them are equal to zero within the limits of permissible error.

Another approach to the problem consists in preliminary rejection in each column of those terms which are obviously equal to zero. But probably the main advantage of this approach is not that it obviates the computation of a certain number of products, but that it permits the computations to be presented in a form more suitable for programming. The most important problem in this case is /659
finding which terms can be neglected. If we assume that the orders of b_i and c_i relative to m are equal to $|2i|$ in the cases of series in even powers of ζ and $|2i - 1|$ in the case of series in odd powers, then in the tables of products the nonunderscored terms have an order of at least ten (product in even powers of ζ) or nine (product in odd powers of ζ). For the satellites VI, VIII, and X of Jupiter, m is near 0.07. Consequently, in the case of an ideal distribution of orders, there can be no doubt that the nonunderscored terms are negligibly small. In real situations, the orders of b_i and c_i can depart from our assumptions. Moreover, the coefficients of the lowest terms in the expansions of b_i and c_i in powers of m have a tendency to increase with increasing orders of the inequalities, so that the product of b_i and c_k can prove to be much larger than would be expected on the basis of its order. Therefore, the figures cited above are only capable at best of giving an approximate idea as to the magnitude of the rejected terms.

On the other hand, it is at once apparent that the unfavorable factors affecting the magnitude of the rejected terms begin to be noticeable only with increasing orders of the inequalities being determined, whereas the required accuracy of the computations in this case is lowered, Consequently, in this case the two processes must compensate one another up to a certain power. Examination of some of the most unfavorable cases reveals that in keeping only the underscored terms we can be assured of the accuracy required for the satellites VI, VII, and X of Jupiter. In the working program, mainly for experimental purposes, a so-called multiplication check block is included. In this block, the sum of the coefficients of the product of a pair of complexes is compared with the product of the sums of the coefficients of the factors. If a term with a large value is not taken into account, the multiplication check block shows this up almost without fail. In any case, it may be noted that the permissible error can be reduced to zero by broadening the limits of variation of the index i.

Let us now see what properties of the product can be drawn upon in formulating the program for multiplication of the groups of coefficients.

Considering the expressions for the coefficients of the product series, or, more correctly, those parts which are summed from the underscored terms, we note the following:

1. In going from one component of each coefficient to the next, the index of the factor b decrements once. The corresponding index of the factor c increments once in type 1 multiplication, in type 2 it decrements once.

2. In going from the last component in the preceding coefficient to the first component in the next coefficient, the index of the factor b increments four times. The corresponding index of the factor c decrements three times in type 1 multiplication, in type 2 it increments three times.

3. The number of components in the coefficients associated with consecutive powers of ζ are alternately equal to four or five.

4. The parameter i for the product series varies within the limits $-4 + \delta \leqslant i \leqslant + 4$, where δ is the result of negating the equivalence of the parity indices of the multiplied inequalities.

6. The index of the factor b in the first component of the coefficient with the largest negative index is always equal to zero. The corresponding index of the factor c in the type 1 is equal to $-4 + \epsilon$ in type 1 multiplication (see 5, above), in type 2 it is equal to $4 - \gamma$, where γ is equal to 1 when the parity index of the first factor is equal to 1 and the parity index of the second factor is equal to 0, and is equal to zero in all other cases.

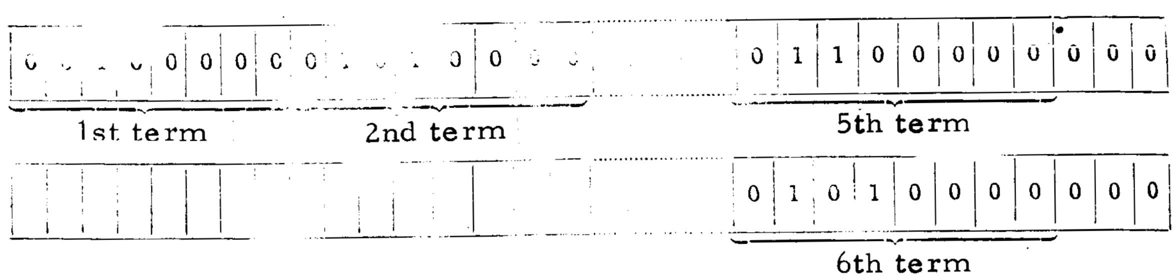
In the expressions for the coefficients of the product series, we can pick out groups of terms other than those above. If we recognize all terms of at least twentieth order (assuming that b_i and c_i are of order m^{21}), the coefficients of the product will be summed from five or six, rather than four or five, terms. The pattern of formation of the coefficient could have been described in terms of properties analogous to those just cited. The magnitude of /660 the error permitted in computing the individual coefficients would be even less in this case. But we will not consider this any further.

The methods indicated permit the formulation of a group multiplication program having only the following at the input: 1) the number of initial cells of groups to be multiplied; 2) the parity indices of the series; 3) multiplication type code.

In our problem, the sequence of series multiplication types is determined by the term of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ subjected to partitioning at this stage. Let this be, for example, $\left(\frac{u_s}{\ }\right)^2\left(\frac{s}{a}\right)$. Whatever the partition, we must first multiply a complex of inequalities of the coordinate u by a complex of inequalities of the coordinate u (type 1 multiplication); then the product, which is always in ascending powers of ζ, must be multiplied by a complex of inequalities of the coordinate s (type 2 multiplication). It is apparent that the type of multiplication for all combinations of groups of a multiplied pair of complexes will remain the same (the case of the coordinate z will be considered separately).

If we agree to represent the types 1, 2, 3, and 4 multiplications by codes 01, 10, 11, 00, then the code for the types of multiplications of three complexes will be 0110 in the investigated example. In general, if the sequence of multiplication types is coded, the first pair of digits is the multiplication type code for the first two complexes, the next pairs of digits give the multiplication type codes for the product of the preceding complexes by the next complex. In the expansions (12) and (13), we encounter partitioning into more than three factors. We agree, however, to allocate eight places to coding of the sequences of multiplication types in individual terms, filling the remaining places with zeros.

The scale of multiplication types of the third row in this case has the form



Since one memory cell of the Strela computer has only 43 places, the logical scale takes up two cells. In the second cell, the terms are not ordered from left to right, but from right to left. We note one fact that will be cause for concern in careful inspection of the scale: The multiplication type code is written on the assumption that the factors in each term are multiplied in order from right to left, rather than left to right. The reason for this is that it is more convenient to work with the factors into which the characteristic is partitioned in just the opposite order, since the order number of the characteristic of the next factor is always located in the reference cell $\langle j \rangle$ (see section 5).

The type of multiplication of the next pair of complexes is segregated from the scale by means of a scanner probe. In going over to multiplications of the next pair of complexes, the logical scale (first cell) is shifted two places to the left. The scale operations are described in further detail in section 15.

Let us now consider how the parity indices and initial cells of the multiplied complexes can be established.

As already mentioned, the machine memory stores the characteristics of the complexes of inequalities. Each of these characteristics $\lambda = \mathbf{e}^{i_1} e'^{i_2} \mathbf{k}^{i_3} \alpha^{i_4}$ is written in memory in terms of the power exponents of the parameters appearing in the characteristic.

The indices i_4, i_2, i_1 are written in the first, second, and third addres- /661
ses of the first cell, i_3 is written in the first address of the second cell allocated for the given characteristic. In the second address of this cell is written the number of the initial cell of the complex having the given characteristic. This pair of cells may be represented schematically as follows:

i_4	i_2	i_1

i_3	N_0	

The number of the initial cells of the complexes can be easily computed prior to the start of operation, since the number of groups in each complex is well known, and to each group is allocated nine cells.

The characteristics are arranged in memory in accordance with their increasing order, beginning with zero order and ending with the highest order characteristics to be taken into account.

If the parameter α enters into the characteristic of a complex in an even power (i_4 even), all groups of this complex will be in even powers of ζ. In case i_4 is odd, the groups are in odd powers of ζ. In view of this, the parity indices of the complexes can be obtained by invoking the last place of the binary representation of i_4 in the characteristics of these complexes. It is obvious that this can be very simply done, provided the order numbers of the characteristics are known. The parity index of the product is obtained as the result of negating the equivalence of the parity indices of the factors.

Knowing the order numbers of the characteristics of the multiplied complexes, we can invoke the numbers N_0 and N_0' of the initial cells of the factors. The address of the initial cells of the multiplied groups can be calculated from one of the combinations of equations (15) and (17), depending on the type of multiplication.

9. COMPUTATION OF THE ARGUMENTS OF THE GROUPS

In multiplying the complexes of inequalities, we run up against the need for adduction of like terms. Let it be required, for example, to multiply $\frac{u_{e^2}\zeta^{-1}}{a} \cdot \frac{u_e\zeta^{-1}}{a}$:

$$\frac{u_{e^2}\zeta^{-1}}{a} \cdot \frac{u_e\zeta^{-1}}{a} = e^2 \sum [\lambda_{i,2c}\zeta^{2i+2c} + \nu_{i,0}\zeta^{2i} + \lambda_{i,-2c}\zeta^{2i-2c}] \cdot e \sum [\mu_{i,c}\zeta^{2i+c} + \mu_{i,-c}\zeta^{2i-c}] =$$

$$= e^3 \Big[\sum_i \sum_k \lambda_{i-k,2c}\mu_{k,c}\zeta^{2i+3c} + \sum_i \sum_k \nu_{i-k,0}\mu_{k,c}\zeta^{2i+c} + \sum_i \sum_k \lambda_{i-k,-2c}\mu_{k,c}\zeta^{2i-c} +$$

$$+ \sum_i \sum_k \lambda_{i-k,2c}\mu_{k,-c}\zeta^{2i+c} + \sum_i \sum_k \nu_{i-k,0}\mu_{i,-c}\zeta^{2i-c} + \sum_i \sum_k \lambda_{i-k,-2c}\mu_{k,-c}\zeta^{2i-3c} \Big].$$

The like terms in the product are those with identical power exponents of ζ in the second and third groups, in the third and fifth groups. In order for the computer to be able to adduce like terms, it must operate not only with the coefficients, but also with the arguments of the groups. Like the characteristics of the complexes, it is impossible to store the arguments of the groups in memory due to the large number of arguments. It is necessary, therefore, before multiplying each pair of complexes, to compute the arguments of their groups anew. Equation (5) is extremely useful in this respect.

To compute the arguments of the groups of a complex, its characteristic is /662 transferred to the input cells of the block for computation of the arguments. The operational flow diagram of this block can be represented in the following form:

$$\downarrow A_1 O_2(k) O_3(i_3') \downarrow O_4(i_2') \downarrow O_5(i_1') \downarrow A_6(i_1', i_2', i_3') F_7(k) T_8 F_9(i_1') P_{10}(i_1' > i_1) \uparrow F_{11}(i_2') P_{12}(i_2' > i_2) \uparrow F_{13}(i_3') P_{14}(i_3' > i_3) \uparrow E$$

The operator A_1 picks out the indices i_1, i_2, i_3 and represents them as integers.

O_2, O_3, O_4, O_5 renew the initial (zero) values of the parameters indicated in the parentheses.

A_6 makes a count according to the equation

$$\tau = (i_1 - 2i_1')c + (i_2 - 2i_2')m + (i_3 - 2i_3')g.$$

F_7, F_9, F_{11}, F_{13} increment the values of the indicated parameters once.

T_8 transmits τ_k to the cell numbered $p_0 + k$.

P_{10}, P_{12}, P_{14} test for the end of the cycles.

It is readily seen that the given computational scheme provides for output of the arguments in the order such that the number of the argument $-\tau_k$ is equal to $K - (k - 1)$, where K is the number of all arguments, k is the order number of the argument $+\tau_k$.

Thus, as a result of the operation of the block, in the cells $p_0 + 1$, $P_0 + 2$, ..., $p_0 + K$ will be obtained the arguments of all groups of a given complex and, in a certain cell <K>, the number of groups of the complex.

10. BLOCK FOR DETERMINATION OF THE INITIAL CELLS

As a result of the operation of the partition block, the cells $<j>$, $<j_1>$, $<j_2>$,..., $<j_{n-1}>$ contain n numbers, which are the order numbers of the characteristics of those complexes which are to be multiplied. In the present block, the first of these numbers is first picked out and, from it, the initial cell and parity index of the first factor are determined. It can happen in this case that the coefficients of the complex are written on tape rather than in internal memory. In this case, they are transferred into a special bank in the operational memory. Into the reference cells are also transferred the arguments of the groups of the complex and their number, computed by the block for determining the arguments. The same quantities are then determined for the second factor. Since the operational memory stores the coefficients of the inequalities of zero, first, and second order and some of the third-order inequalities, the coefficients of no more than one factor need be transferred from tape. Also, in this same block is performed a computation of the quantities ϵ and δ (see section 8), after which control is transferred to the block for determining the addresses of the products. After multiplication of the first pair of complexes, control is again transferred to the block for determination of the initial cells, which calls into operation the third factor with characteristic number j_2, all the quantities for this factor pair up with the corresponding quantities obtained for generation of the preceding factors, and so on until all j_k are exhausted.

The logical diagram for this block can be written in the form

$$\overset{IX\,20;\,XI\,19}{\downarrow} T_1 \overset{I\cdot 11}{\downarrow} O_2(\tilde{j})\, O_3 \overset{VII\,13}{\downarrow} \uparrow F_4(\tilde{j})\, A_5(\tilde{j})\, E_{1_6}\, P_7 \,|\, E_8 \,|\, \boxed{T_9} \downarrow P_{10}(\tilde{j}<2) \,|\, 3_{11} \,|;\, \uparrow T_{12} \,|;\, \uparrow P_{13}(\tilde{j}<2) \,|\, T_{14} \downarrow T_{15} \,|;\, \downarrow T_{16} \downarrow T_{17}\, A_{18} \overset{IV\,1}{\uparrow}$$

The operator T_1 transmits an instruction as to by-passing of the operator /663 T_9 in repeated use of one partition of the characteristic. It is essential for the shortening of machine time.

O_2 renews the initial value of the parameter $\tilde{j}$ ($\tilde{j} = 0$). $\tilde{j}$ is the number of the factor.

O_3 renews the logical scale by transmitting the contents of the cell g_2 (see section 15) to the cell of the logical scale G.

A_5 selects from the block of characteristics the characteristic of the next factor.

E_{1_e} returns to the subroutine, the block for computation of the arguments.

P_7 tests for fulfillment of the condition $N_0 > 4000$, which is true if the coefficients of the complex are written on tape.

E_8 calls for bypass of the operator T_9 or return to it (see T_1).

T_9 transmits the zone of the magnetic tape with number N_0 to the reference bank of the operational memory.

T_{11} transmits to the cell R, in which is formed one of the commands of the multiplication block, having the form

N_0+9n	$N'_0+9l-4+\varepsilon$	σ	$\times$

or

N_0+9n	$N'_0+9l+4-\gamma$	σ	$\times$

(cf. Φ_{15} of the block for analysis of the multiplication types), as the first address the number of the initial cell of the reference bank, to which are transmitted from tape the coefficients of the complexes of inequalities.

T_{12} executes the same operation, but with respect to the second address.

T_{14} transmits to the first address of the cell R the number of the initial cell of the first complex to be multiplied.

T_{16} transmits to the second address of the cell R the number of the initial cell of the second complex to be multiplied.

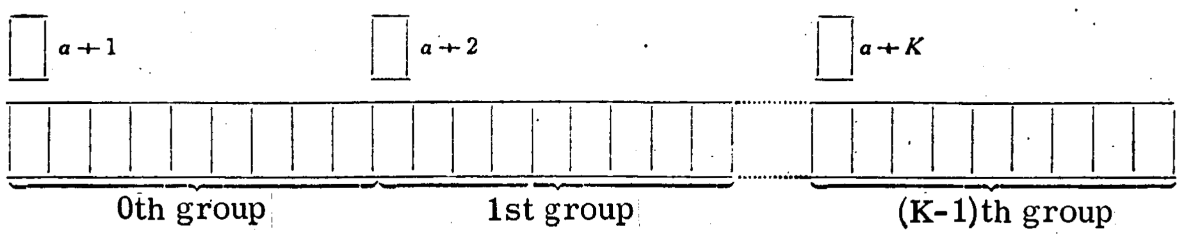
T_{15} and T_{17} segregate the parity indices of the multiplied complexes.

A_{18} computes ϵ and δ for the series to be multiplied and certain other auxiliary quantities.

11. BLOCK FOR DETERMINATION OF THE PRODUCT ADDRESSES

In multiplying the complexes, we will adhere to the following order of operation: all groups in the order of their numbers are first multiplied by the first group of the second complex, then in the same order they are multiplied by the second group of the second complex, etc.

We send the products of all the complexes to the portion of memory intended for this purpose, the product bank. For the coefficients of a product of two groups of terms, ten cells of this bank are allocated. Furthermore, one cell is set aside for writing in the argument of the product. The product bank must be as many groups in each of the ten cells as there are different possible arguments τ of the product of complexes. This can be represented schematically as follows:



The numbers of the groups to which the products of each pair of groups of complexes to be multiplied are to be transmitted are computed in the block for determining the addresses of the products. In this block are systematically formed sums of the arguments of the groups: /664

$$\tau_1+\sigma_1,\ \tau_2+\sigma_1,\ \tau_3+\sigma_1, \ldots;\ \tau_1+\sigma_2,\ \tau_2+\sigma_2, \ldots;$$

After formation of the next sum, it is compared with the contents of the cells a + 1, a + 2,... of the product bank (all cells of the bank are cleared prior to computation of the product of each pair of complexes). In the case when $\tau_n + \sigma_\iota$ turns out to be equal to the contents of one of the cells a + 1 + p, the product of the group with number n by the group with number ι is transmitted to the group of cells of the bank with number p (the coefficient of the product d_i is transmitted to the cell with number a + k + 9p + 5 + i). If $\tau_n + \sigma_\iota$ is not equal to the contents of either of the occupied cells a + 1, a + 2,..., then $\tau_n + \sigma_\iota$ is transmitted to the first empty cell a + 1 + r and the product is sent to the group of the bank with number r. Since the program is constructed such that the product coefficients sent to the cells of the product bank are summed with the contents existing therein, the process described here provides for the adduction of like terms.

The logical diagram of the block has the form

$$\overset{II\,18}{\downarrow} O_1(q)\, O_2(l) \uparrow F_3(l)\, P_4(l<L) \lfloor O_5(n) \overset{VII\,10 \;\; VI\,9}{\downarrow} F_6(n)\, P_7(n<N) \lfloor A_8(l,n)\, O_9(p) \downarrow P_{10}(p \neq q) \lfloor P_{11}(a+l+p \neq \tau) \lfloor F_{12}(p) \lceil ; \uparrow F_{13}(q)\, T_{14} \uparrow \Phi_{15} \overset{V\,1}{\lceil}$$

The operators O_1, O_2, and O_3 renew the initial values of the parameters q ($q = 0$), ι ($\iota = -1$), and n ($n = -1$); q is interpreted as the number of occupied groups of the product bank.

A_8 forms the sum $\tau_\iota + \tau_n = \tau$.

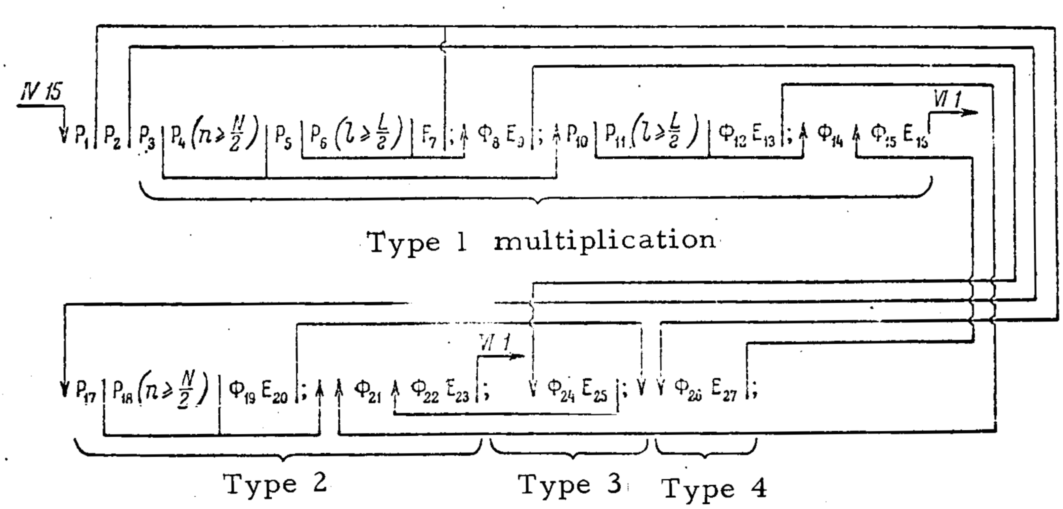
T_{14} transmits $\tau_\iota + \tau_n$ to the cell $a + 1 + p$.

Φ_{15} generates for the multiplication block the command to transfer the product to cells of the group with number p.

12. BLOCK FOR ANALYSIS OF THE TYPE OF GROUP MULTIPLICATION

After operation of the block for determining the addresses of the products, control is acquired by the block for analysis of the type of group multiplication. Part of the function of the block is to form a series of commands for the multiplication block, depending on the type of series multiplication. These commands cannot be generated once and for all for all combinations of groups of a multiplied pair of complexes, since the type of multiplication can change as the numbers of the groups varies, even if only one of the multiplied complexes is a complex of inequalities in the coordinate z. In fact, the form of equations (18) and (18a) for complexes of inequalities in the coordinate z shows that in the first half of all the groups, the series are in ascending powers of ζ, while in the second half of the groups the series are in descending powers of ζ. Consequently, if the type of multiplication determined from the scale is the first type, corresponding to multiplication of two series in ascending powers of ζ (for example, $z_{\mu} \sqrt{-1} \cdot u_{\lambda} \zeta^{-1}$), then all the pairs of groups for which $n < N/2$ must be multiplied in correspondence with the first type of multiplication, and all pairs of groups for which $n \geq N/2$ must be multiplied in correspondence with the third type of multiplication (multiplication of a series in descending powers by a series in ascending powers of ζ).

A more complete analysis in this type of multiplication must be carried out with allowance for the fact that the second factor can be a complex of inequalities in the coordinate z. Thus, the type of multiplication of individual pairs of groups depends, first of all, on the type of multiplication of the complexes as specified by the logical scale, secondly, on the affiliation of these groups with complexes of inequalities of the coordinate z, and thirdly, on the order /665 numbers of the groups. An analysis of the possible situations can be made according to the following scheme:



The operators P_1 and P_2 determine the type of multiplication specified by the logical scale and, in correspondence with it, transfer control to the various groups of operators. It is readily seen that the factors in the terms of expansion of $\partial\Omega/\partial s$ and $\partial\Omega/\partial z$ can be arranged in an order such that type 3 multiplication is avoided. For this reason, the diagram does not transfer control from P_1 and P_2 to the group of operators controlling type 3 multiplication.

The group of operators designated from type 1 multiplication analyzes all possible cases which can occur in this type of multiplication. The same applies to the group of operators designated for type 2 multiplication. Types 3 and 4 are not in need of such an analysis, because these multiplication types do not occur for complexes of inequalities in the coordinate z. It is found as a result of analysis that the given pair of groups should be multiplied in accordance with some other type of multiplication, the operators E_7, E_9, E_{13}, E_{20} transfer control to the group of operators for the corresponding type, which generate commands for the multiplication block. These operators in the first type of multiplication are Φ_{14} and Φ_{15}, in the second type Φ_{21} and Φ_{22}, in the third type Φ_{24}, and in the fourth type Φ_{26}. Since, as mentioned, the coefficients of ζ^{-2i} in types 3 and 4 multiplication are respectively equal to the coefficients of ζ^{2i} in types 2 and 1 multiplication, Φ_{24} and Φ_{26} only execute a part of the necessary operations, all the rest being carried out by the operators Φ_{22} and Φ_{15}, to which control is transferred.

P_3 and P_{17} test fulfillment of the following condition: The parameter k enters into the characteristic of the first factor in an odd power (in this case, the first complex is a complex of inequalities in the coordinate z). P_5 and P_{10} test the second factor in the same manner.

Φ_{14} calculates the number of the initial cells of the groups to be multiplied, on the basis of equation (15):

$$N = N_0 + 9n \quad \text{и} \quad N' = N_0' + 9l,$$

the quantities appearing in the first and second addresses of the cell R being used for N_0 and N_0' (cf. T_{11} in the block for determining the initial cells). Inasmuch as n and ι are counted from zero, n - 1 and ι - 1 are replaced in equation (15) by n and ι. Moreover, Φ_{14} transmits to the reference cells the address substitution constants, in such a predetermined manner that the product coefficients d_i will be transmitted with increasing index i into group cells of the product bank in order of increasing cell number.

Φ_{15}, consistent with the property 6 of the product series coefficients, forms the quantity $N_0' + 9\iota - 4 + \epsilon$ in the second address of the cell R; that quantity is equal to the address of the second factor in the expression $b_{i-k}c_k$ /666 with the initial values of the indices i and k (the address of the first factor is equal to $N_0 + 9n$). Besides this, Φ_{15} transmits the address substitution constants in correspondence with properties 1 and 2.

The corresponding operators for the other types of multiplication execute an analogous function. Φ_{24} and Φ_{26} transmit the address substitution constants in such fashion that the coefficients d_i are transmitted with increasing i into group cells of the product block in order of diminishing cell number, beginning with the highest numbered cell. In this way, the third and fourth types of multiplication are reduced, on the basis of the remarks made on page 31, to the second and first types of multiplication, respectively. The operators Φ_8, Φ_{12}, and Φ_{19} generate the command to change sign of the product $b_{i-k}c_k$ or $b_{i-k}c_{-k}$, since in these cases one of the factors is taken from memory with the opposite sign (see section 6).

13. MULTIPLICATION BLOCK; MULTIPLICATION CHECK BLOCK

The logical diagram of the multiplication block can be written as follows (operators A_1 - E_9):

$$\overset{V\,15;\,V\,23}{\downarrow} A_1(i,k)\,A_2\,A_3(i)\,F_4(k)\,P_5\,\Phi_6\,F_7(i)\,P_8\,E_9 \xrightarrow{I\,6} \overset{IV\,4}{\downarrow} P_{10}\,Z_{11};\ P_{12}(\tilde{j} \leqslant \tilde{J})\,T_{13}\,E_{14} \xrightarrow{VIII\,1} \xrightarrow{II\,4}.$$

The operator A_1 executes multiplication of $b_{i-k}c_k$ (or $b_{i-k}c_{-k}$).

A_2 either leaves the product unaltered or changes the sign (cf. Φ_8, Φ_{12}, Φ_{19} of the preceding block).

A_3 sums the result of the preceding operations with the contents of the cell $a + K + 9p + 5 \pm i$ (possibly also $a + K + 9p + 6 - i$).

F_4 performs an address substitution in k in correspondence with property 1 of the product series coefficients.

P_5 checks the transfer to computation of the next coefficient in fulfillment of the following condition: The number of components in the coefficient is greater than $\tilde{k}$, where the initial value of $\tilde{k}$ is computed in accordance with property 5.

Φ_6 forms the new value of $\tilde{k}$ in correspondence with property 3.

F_7 performs an address substitution in i of the operators A_1 and A_3.

P_8 checks fulfillment of the condition $i \geqslant 5$.

E_9 transfers control to the operator F_6 of the block for determining the product addresses.

After all n and i have been exhausted in the block for determining the product addresses, meaning that multiplication of the pair of complexes has been completed, control is transferred to the multiplication check block (P_{10} and

Z_{11} of the preceding diagram). The multiplication check block is not a fixed element of the program. As already stated, the operator P_{10} sums all coefficients of the product of a pair of complexes and compares them with the product of the control sums of the factors (the control sum is the sum of all coefficients of a complex of inequalities; the control sum is stored in memory along with the coefficients of the complex). In the event that these quantities do not coincide, the operator Z_{11} executes a stop.

P_1 checks the need for returning to the block for determination of the initial cells if the number of factors is insufficient. In this case, the operator T_{13} transfers the product series from the product bank to a special location in memory; it transmits to the first address of the cell R (cf. T_{11} of the block for determination of initial cells) as N_0 the number of the initial cell of the bank to which the product series is transmitted; it transmits to the reference cells the parity index and number of arguments of the product; it shifts the contents of the cell G of the multiplication type scale two places to the left. Since the product is always in ascending powers of ζ, the quantity i_3 is set equal to zero (this is necessary in order for the operators P_3 and P_{17} of the multiplication type analysis block to function properly).

If all the necessary complexes have been multiplied, control is transferred [667] to the next block (below).

14. BLOCK FOR MULTIPLICATION BY K $\zeta^{\beta} a_f^*$

This block executes on the product series all the rest of the operations stipulated by the form of the term in the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$, including multiplication by the coefficient K, multiplication by ζ^{β}, and multiplication by one of the series $\bar{a}_2^*$, b_2^*, $\bar{a}_3^*$, ... Let, for example, the products with characteristic λ generated by the term $\frac{15}{8}\left(\frac{s\zeta_1}{a}\right)^2 \zeta^{-3}\bar{a}_3$ of the expansion of $\partial\Omega/\partial s$

be computed. Then, having executed multiplication of the complexes $\frac{s_\mu \zeta}{a} \cdot \overline{\frac{s_\nu \zeta}{a}}$,

for the next partition of the characteristic $\lambda_1 = \mu\upsilon$, the product must be multiplied by $\frac{15}{8} \zeta^{-3} \bar{a}_3^*$, where $\bar{a}_3^*$ denotes either 1, or $(5\zeta^m - \zeta^{-m})$, or

$\zeta^{2m} \ldots - \frac{\ldots}{8}\zeta^{-2m})$, depending on what power of e' the initial characteristic λ/α for the second row is divided by in a given cycle of computations.

For a graphic illustration of how the block functions, we examine the ordering of the coefficients of the expansions in machine memory. The coefficients of the expansions of $\partial\Omega/\partial s$ and $\partial\Omega/\partial z$ are written row by row in successive cells of the memory. The coefficients of the expansions $\bar{a}_2$, b_2, $\bar{a}_3$,... can be arranged in tabular form:

		e'^0	e'^1		e'^2			e'^3			
		ζ^0	ζ^m	ζ^{-m}	ζ^{2m}	ζ^0	ζ^{-2m}	ζ^{3m}	ζ^m	ζ^{-m}	ζ^{-3m}
1	$\bar{a}_2$	1 0	13 $\frac{7}{2}$	25 $-\frac{1}{2}$	37 $\frac{17}{2}$	49 $-\frac{5}{2}$	61 [illegible]	73 $\frac{845}{48}$	85 $-\frac{123}{16}$	97 $\frac{1}{16}$	109 $\frac{1}{48}$
2	b_2	2 0	14 $\frac{3}{2}$	26 $\frac{3}{2}$	38 $\frac{9}{4}$	50 $\frac{5}{2}$	62 $\frac{9}{4}$	74 $\frac{53}{16}$	86 $\frac{27}{16}$	98 $\frac{27}{16}$	110 $\frac{53}{16}$
3	$\bar{a}_3$	3 1	15 5	27 -1	39 $\frac{127}{8}$	51 -6	63 $\frac{[illegible]}{8}$	75	87	99	111
4	c_3	4 1	16 1	28 3	40 $\frac{11}{8}$	52 2	64 $\frac{53}{8}$	76	88	100	112
5	$\bar{c}_3$	5 1	17 3	29 1	41 $\frac{53}{8}$	53 2	65 $\frac{[illegible]}{8}$	77	89	101	113
6	$\bar{c}_3$	6 1	18 3	30 1	42 $\frac{53}{8}$	54 2	66 [illegible]	78	90	102	114
7	$\bar{a}_4$	7 1	19 $\frac{13}{2}$	31 $-\frac{3}{2}$	43	55	67	79	91	103	115
8	c_4	8 1	20 $\frac{1}{2}$	32 $\frac{9}{2}$	44	56	68	80	92	104	116
9	$\bar{c}_4$	9 1	21 $\frac{9}{2}$	33 $\frac{1}{2}$	45	57	69	81	93	105	117
10	h_4	10 1	22 $\frac{5}{2}$	34 $\frac{5}{2}$	46	58	70	82	94	106	118
11	$\bar{c}_4$	11 1	23 $\frac{9}{2}$	35 $\frac{1}{2}$	47	59	71	83	95	107	119
12	b_4	12 1	24 $\frac{5}{2}$	36 $\frac{5}{2}$	48	60	72	84	96	108	120

In this table, $\bar{a}_2$, b_2, $\bar{a}_3$,... are numbered in the order in which they are /668
encountered in the terms of the expansion of $\partial\Omega/\partial s$. For all coefficients, the relative numbers (i.e., the numbers counted from some arbitrary origin) of the cells in which they are stored are indicated. At the top of each column is the power of ζ, which is multiplied by each coefficient of that column. The horizontal braces join the columns in which one row contains coefficients of the terms of the expansions of $\bar{a}_2$, b_2, $\bar{a}_3$,... having the indicated power of e' as a factor.

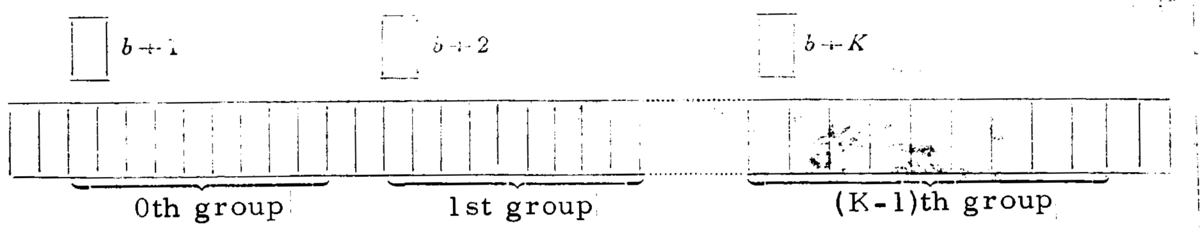
An analogous table can be drawn up for the expansion of $\partial\Omega/\partial z$.

Each of the quantities a^*_f will be completely defined if we can indicate the number of the cell in which the coefficient of its first term is stored, the value of the exponent of ζ in this term, and the number of all terms. All of these quantities can be easily computed if the number f of the term of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ on which the computation is based at the given instant is known, along with the power exponent h of e' by which the initial characteristic for the working row is divided in a given cycle of computations. Then, as readily visualized in the table, the first coefficient a^*_f will have the relative number

$$f - 12 \sum_{k=1} k,$$

the exponent of ζ in this term is equal to mh and the number of terms is equal to h + 1. In going from the first term to the second, the address of the coefficient is increased 12 units, the power exponent of ζ is deminished by 2m, etc.

To write the right-hand members of the fundamental equations (7) and (8), a bank of cells is allocated, as can be schematically represented in the following form:



The arguments of the groups of the sought-after complex of inequalities, $\tau_1, \tau_2, \ldots,$ are transmitted to the cells b + 1, b + 2 prior to the start of computation of the right-hand member.

At the instant control is transferred to the block for multiplication by $K\zeta^{\beta}a^{*}_{f}$, the result of multiplication of the complexes of inequalities is located in cells of the product bank. Subsequently, all occupied groups of the product bank must be first multiplied by the first term of a^{*}_{f}, then by the second, third, and so on, until all terms are exhausted. The product of each group by a definite term of a^{*}_{f} must be multiplied by $K\zeta^{\beta}$ and sent to the same groups of the bank of cells for the right-hand member whose argument coincides with the product argument (in actuality, the modulus of the difference in arguments is evaluated, which must be less than or equal to some maximum admissible computational error). Of course, the newly transmitted terms must be summed with the already existing contents of the cells in the bank for the right-hand member (the bank is cleared before starting computation of the right-hand member). This process of adduction of like terms follows the same plan as in the address determination block. Let it be required, for example, to multiply by $-\frac{15}{8}\zeta^{-3}(5\zeta^{m}-\zeta^{-m})$. First of all, the first of the product series arguments found in the cell a + 1 of the product bank is summed with the argument m of the first term of $\bar{a}^{*}_{3}$. The sum is then compared with the contents of the cells for the arguments in the right-hand member bank, b + 1, b + 2,... We assume that a /669 match has occurred in the pth step (p counted from zero). Then after multiplication by $-\frac{15}{8} \cdot 5\zeta^{-3}$, the first group of the product bank is transmitted to cells of the group with number p of the right-hand member bank. This transfer, however, is not allowed to go such that the contents of the cell with relative number i of the first group will fall into a cell with the same relative number of the second group; since the series is multiplied by ζ^{-3}, this transfer must be accompanied by a certain shift. Then this cycle of computations is repeated with groups whose arguments are located in the cells a + 2, a + 3,... After exhausting all groups, a transfer is executed to the term $-\zeta^{-m}$ of the expansion $\bar{a}_3^*$ for repetition with it of the next broader cycle.

We will now deduce a formula for the magnitude of the shift.

It is readily seen that multiplication of a series in powers of ζ by ζ^2 must be accompanied by a transfer of the coefficient λ_i of ζ^{2i} to the cell numbered one unit higher than the number of the cell in which λ_i is stored, because now λ_i becomes the coefficient of $\zeta^{2(i+1)}$. Similarly, in multiplying the series by ζ^{-2}, the coefficients of the series are shifted one cell to the left. However, the magnitude of the shift depends not only on the power of ζ, by which the series is multiplied, but also on the mutual relationship of the parity indices of the product series and the computed right-hand member or, equivalently, the parity index of the complex of inequalities whose coefficient are being determined. The following four cases are possible:

1) Parity index of the product equal to 0, parity index of the right-hand member equal to 0.

Both series are in even powers of ζ. In multiplication of the product series by ζ^{β}, the shift is equal to $\beta/2$.

2) Parity index of the product equal to 1, parity index of the right-hand member equal to 0.

Multiplication of the product series by ζ^{β} can be represented as multiplication by $\zeta^{\beta-1} \cdot \zeta$. But multiplication of the product series by ζ reduces to transformation of this series from a series in odd powers of ζ to a series in even powers, which is what it should be by the fact that the parity index of the right-hand member is zero. The magnitude of the shift is equal to $\frac{1}{2}(\beta - 1)$.

3) Parity index of the product equal to 0, parity index of the right-hand member equal to 1.

Multiplication of the product series by ζ^{β} can be represented as multiplication by $\zeta^{\beta+1} \cdot \zeta^{-1}$. Multiplication by ζ^{-1} reduces to transformation of the product to a series in odd powers of ζ. The shift of the product series is equal to $\frac{1}{2}(\beta + 1)$.

4) Parity index of the product equal to 1, parity index of the right-hand member equal to 1.

Both series are in odd powers of ζ. In multiplication by ζ^{β}, the shift is equal to $\beta/2$.

All possible cases are covered by the following formula:

$$\text{Shift} = \frac{\beta + \text{parity index of right-hand member} - \text{parity index of product}}{2}$$

The expansions (12) and (13) show that for those rows for which β is even, the difference in parity indices is equal to 0, for those rows for which β is odd, the difference is equal to ± 1. Consequently, the magnitude of the shift is an integer.

It is readily calculated that the largest possible shift for the expansions (12) and (13) is equal to 2. When the shift is made, one or two of the coefficients will run over the nine cells allotted for the coefficients of one group of terms. So that these coefficients will not get into the cells of neighboring groups, each group of the right-hand member bank is set off from the other by two cells. The terms which run out of the group are disregarded /670 from then on. This is fully justified by the considerations of section 6. Over and beyond the arguments therein, we point out that all the terms of the right-hand member that are generated by $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ are multiplied by the small factor m^2. Moreover, the maximum shift of two cells occurs in determining the coefficients of inequalities whose characteristics include the very small quantity α^2.

The logical diagram of the block for multiplication by $K \overset{m}{a}{}^*_f$ has the form

$\xrightarrow{VII\,12} \downarrow A_1\, O_2(t) \downarrow A_3(f,t)\, O_4(\iota) \downarrow O_5(p)\, A_6(t,\iota) \downarrow P_7\, F_8(p)\, P_9(p$ [illegible] Z [illegible] $(\iota, t, p)\, F_{12}(\iota)\, P_{13}(\iota \geq q)\, F_{14}(t)\, P_{15}(t > h)\, E_{16} \xrightarrow{IX\,16;\,XI\,15}$

The operator A_1 computes the magnitude of the shift.

O_2, O_4, O_5 renews the initial values of the parameters t $(t = 0)$, ι $(\iota = 0)$, and p $(p = 0)$.

A_3 computes the product of the coefficient of the expansion of $\partial\Omega/\partial s$ with number f by the coefficient of the expansion of a^*_f located in the cell with relative number equal to $f + 12\left(t + \sum_{k=1}^{k=h} k\right) \equiv T_\iota$; it computes the value of the argument of the corresponding term in the expansion of a^*_f, equal to $m(h - 2t) \equiv S_t$.

A_6 forms the sum of the argument S_t and the argument of the group in the product bank with number ι.

P_7 tests fulfillment of the following condition: The sum of the arguments as formed by the operator A_6 is not equal an argument from the cell with number $b + 1 + p$ of the right-hand member cell bank.

P_9 tests the following condition: $p > P$, where P is the number of all groups of the complex of inequalities being determined. In case the condition is fulfilled, the operator Z_{10} executes a stop, since fulfillment of the condition means a halt in computer operation.

A_{11} performs multiplication of terms of the group in the product bank with number ι by the coefficient T_t and transmits them to the group of the right-hand member bank with number p, simultaneously executing a shift by the required number of cells.

P_{13} tests the condition $\iota \geqslant q$, where q is the number of occupied groups in the product bank.

E_{16} can transfer control to one of the executive blocks, depending on how this operator is formed by the executive block.

15. EXECUTIVE BLOCK FOR COMPUTATION OF TERMS IN THE RIGHT-HAND MEMBER OF EQUATIONS (7) OR (8), GENERATED BY $\partial\Omega/\partial s$ OR $\partial\Omega/\partial z$

The function of the executive block can be succinctly described as control of the program cycles. Inasmuch as the number of cycles is rather large, it is convenient for simplication of the logical diagram for the program to assemble the control operators for part of the cycles in one memory location and to regard the set of them as a program block. This block contains the operators for control of those cycles within which transitions are made from one term of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ to another. There are five such cycles.

Cycle 1. Computation of terms with a definite characteristic, generated on the basis of one partition of the characteristic of a group of row terms having the same type of partition. The working part of the cycle embraces blocks II-VII. The cycle control operators (O_{11} - E_{15}, F_{16} - E_{20}; see the diagram) execute multiple repetition of the operation of blocks II-VIII for the successive terms of the group.

Cycle 2. Computation of terms with a definite characteristic, generated on the basis of all admissible partitions of the characteristic of a group of row terms having the same type of partition. The program of cycle 2 embraces the program of cycle 1, including in addition to it the partition block. The control operators for this cycle (T_{21} - E_{23}) execute multiple repetition of cycle 1 every time on the basis of a new partition of the characteristic. The end-control operator for this cycle is left in the partition block (see P_{16} of the partition block).

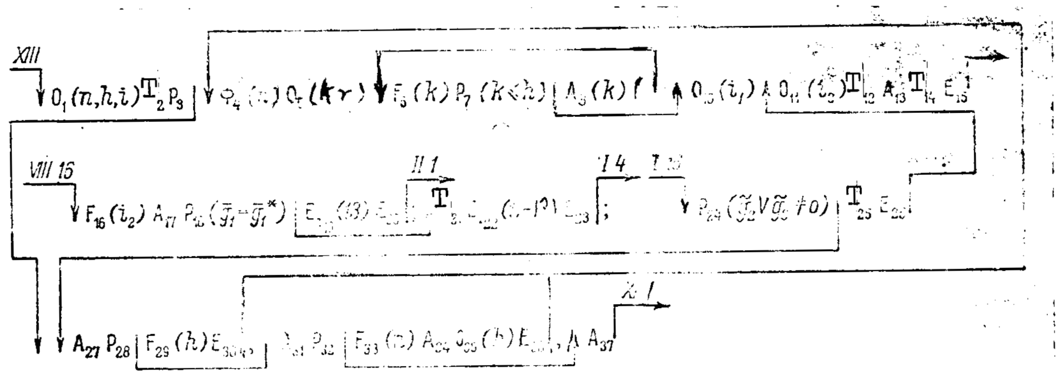
Cycle 3. Computation of terms with a definite characteristic, generated by all groups of terms of one row with a definite value of the parameter h. The control operators for the cycle (P_{24} - E_{26}) execute repetition of the preceding cycle, performing address substitution on it each time to a new group of terms in the row.

Cycle 4. Computation of terms with characteristic λ, generated by the row with number n for all possible values of the parameter h for that λ. The control operators for the cycle are A_{27} - E_{30}.

Cycle 5. Cycle in the parameter n. The control operators for the cycle are A_{31} - E_{36}.

It was stated in section 14 that it is necessary for operation of the block for multiplication by $K\zeta^{\beta}a_f^*$ to know f, the number of the working term at a given instant. The value of f can be represented as the sum of three parameters $f - 1 = i + i_1 + i_2$, where i is the number of terms of the expansions in the rows preceding the working row at that time, i_1 is the number of terms of the expansion in the given row before the working group, i_2 is the number of terms in the group before the working term. The representation of f - 1 in this form proves convenient, in that the quantities vary in quite simple fashion for all possible address substitutions and renewals in the five indicated cycles.

Three groups of cells are picked out for address substitution of the logical scales: g_1, g_2, g_3; $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$; $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{g}_3$. To the cell g_1 is transmitted the scale of partition types and to the cells g_2, g_3 the scale of type of multiplication for the row with number n. With respect to address substitutions of the scales, the groups of cells g_j, $\bar{g}_j$, and $\tilde{g}_j$ play a role analogous to that which i, i_1, i_2 play in address substitutions of the quantity f. The diagram of the executive block has the form



The operator O_1 receives control from the executive routine in transition to determination of the coefficient of the next complex of inequalities; it renews the initial values of the parameters n = 1, h = 0, i = 0.

T_2 transmits the characteristic λ to two pairs of reference cells L_1 and L_2 (for brevity, each pair will be designated by a single symbol); it transmits to the operator VII-16, I-16 commands to return to the present block.

P_3 tests the following condition: λ/α does not contain negative powers of α.

Φ_4 generates comparison constants depending on n: $\tilde{\tilde{n}}$ (see the partition block, P_9) and $\tilde{J}$ (see the multiplication check block, P_{12}); it selects from memory the logical scales of the row with number n and transmits them to g_1, g_2, g_3.

A_8 forms the sum $r + 12k$; O_5 - E_9 compute the sum $r = 12 \sum_{k=1}^{k=h} k$, used in 672 the block for multiplication by $K\zeta^{\beta}a_f^*$.

O_{10}, O_{11} renew the initial values of the parameters i_1, i_2 ($i_1 = 0$, $i_2 = 0$).

T_{12} transmits g_j into $\bar{g}_j$.

A_{13} forms the sum $i + i_1 + i_2$.

T_{14} transmits the characteristic (L_1) into the cell D_1 (see the partition block).

F_{16} performs address substitution of i_2 by unity.

A_{17} stores $\bar{g}_1$ in memory in the cell $\bar{g}_1^*$; it shifts the contents of the cell $\bar{g}_1$ 4 places, those of $\bar{g}_2$ 8 places to the left; it augments the free spaces of $\bar{g}_2$ with a group of places of $\bar{g}_3$; it shifts $\bar{g}_3$ 8 places to the right.

P_{18} tests the following condition: $\bar{g}_1 = \bar{g}_1^*$ (fixed type of partition in transition to the next term).

$E_{1_{19}}$ returns to the subprogram - to the operator A_{13}.

T_{21} transmits $\bar{g}_j$ into $\tilde{g}_j$, i_2 into $\tilde{i}_2$. In this way, the position of the scales and value of the parameter i_2 are fixed for the first term of the next group in the row.

$E_{1_{22}}$ returns to the subprogram - to the operators O_{11} - A_{13}.

P_{24} forms the logical sum of the contents of the cells $\tilde{g}_2$ and $\tilde{g}_3$. In the event that this sum is equal to zero (which can happen when all groups of the row have been exhausted), it transfers control to the control operators for the cycle in h.

T_{25} transmits $\tilde{g}_j$ into g_j and replaces i_1 by $i_1 + \tilde{i}_2$ (see T_{21}).

A_{27} forms the characteristic $(L_1)/e'$ and transmits it into the cell L_1.

P_{28} tests the following condition: $(L_1)/e'$ does not contain negative powers of e .

F_{29} performs address substitution in h.

A_{31} forms the characteristic $(L_2)/\alpha$ and transmits it into the cell L_1.

P_{32} tests the following condition: $(L_2)/\alpha$ does not contain negative powers of α.

F_{33} performs address substitution in n.

A_{34} replaces i by $i + i_1 + i_2$.

O_{35} performs renewal (reset) in h.

A_{37} multiplies the contents of the cells of the right-hand member bank by m^2.

16. EXECUTIVE BLOCK FOR COMPUTATION OF TERMS IN THE RIGHT-HAND MEMBER OF EQUATIONS (7) OR (8), GENERATED BY SERIES EXPANSION OF $u\zeta^{-1}/r^3$ OR $z\sqrt{-1}/r^3$

As already mentioned, the program for computation of the right-hand members of the fundamental equations is written such that all that is required for transition from the first computational stage to the second is replacement of the executive block. Let us examine the essential differences in the expansions contained in the brackets in the right-hand members of equations (7) or (8) and the expansions of $\partial\Omega/\partial s$ and $\partial\Omega/\partial z$.

In the terms of the expansions generated by $u\zeta^{-1}/r^3$ or $z\sqrt{-1}/r^3$, there is no multiplication by $\zeta^{\beta}a_f^*$. However, no modification is required in the block for multiplication by $K\zeta^{\beta}a_f^*$, for if we let $\beta \equiv 0$, $h \equiv 0$, the block functions just as well in this case also. The cycle in h vanishes from the executive block.

In the terms of these expansions, the characteristic μ cannot take on values $\mu = 1$ (in the expansions of $\partial\Omega/\partial s$ and $\partial\Omega/\partial z$, the terms $u_0\zeta^{-1}$ and $s_0\zeta$ are included in the expressions $\sum u_\mu \zeta^{\mu-1}$ & $\sum s_\mu \zeta_\mu$; this is not done here). Consequently, other restrictions must be placed on the set of values that the

parameter j_1, the order number of the first factor in the bank of characteristics associated with partition of the characteristic into two parts, can assume. /673
If previously the parameter j_1 varied within limits

$$1 \leqslant j_1 \leqslant J,$$

where J is the number of characteristics whose order is no higher than the order of the characteristic being partitioned, then the parameter j_1 must now vary within the limits

$$2 \leqslant j_1 \leqslant J_1,$$

where J_1 is the number of characteristics whose order is less than the order of the characteristic to be partitioned.

The quantity J (or J_1) is determined in the partition block from a table as a function of the order of the characteristic. It is sufficient, therefore, in the first case to subtract 0 from the order of the characteristic, 1 in the second case, so as to account for this change in the upper limit. The lower limit of variation of j_1 is analogously corrected.

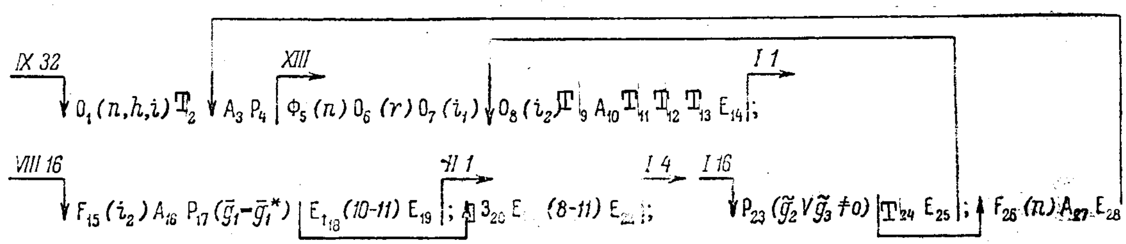
In the expansions generated by $u\zeta^{-1}/r^3$ or $z\sqrt{-1}/r^3$, each term is multiplied by one of the quantities P, Q, R,... or their conjugates. We note first of all that multiplication of the product of complexes of inequalities by the series P, Q, R,... can be executed by the same program as before. All that is necessary to do this is to introduce beforehand into machine memory the coefficients

$$P_i,\ Q_i,\ R_i,\ \ldots\ i = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4, \ldots$$

and the "characteristics" of these "inequalities." It is necessary in the characteristics to set $i_1 = i_2 = i_3 = i_4 = 0$. The number of the initial cell of the inequality must be written in the second address of the second cell of such a characteristic, as in the general case. In the scales of multiplication types for each row, the type of multiplication by these inequalities must be indicated.

As a result of the operation of the partition block, the numbers n of the characteristics are formed in the cells $<j>$, $<j_1>$, $<j_2>$,..., $<j_{n-1}>$. An additional problem that arises in this case is to obtain in the cell $<j_n>$ the number of the characteristic of whichever of the series P, Q, R,... multiplies the expansion of the product of the remaining series in the given term. As indicated, each partition of the characteristic is used for several terms of one row, whereas the multipliers P, Q, R,... are different in general for these terms. Consequently, the address substitution of $<j_n>$ must be performed for each address substitution in the parameter i_2. For this purpose, the numbers of the characteristics of P, Q, R,... are written in successive memory cells in the order in which the quantities P, Q, R,... are encountered in the expansion generated by $u\zeta^{-1}/r^3$ or $z\sqrt{-1}/r^3$. Knowing f (the number of the current working term) and n (row number), it is easy to extract the number of the characteristic from these cells and to send it into the cell $<j_n>$. We assume that j_n is transmitted after the partition block has generated one of the admissible partitions for the initial characteristic (in this case, $i = \tilde{n}$, see the partition block). If the acquisition of each partition were followed by reversion to the executive block, then j_n could always be transmitted in this case as indicated. However, such reversions are not dictated by necessity, at least insofar as the program for selecting terms from the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ is concerned; in this program, after each partition has been obtained, reversion occurs to the block for determination of the initial cells. If we wish to keep the old program for selecting terms from the expansions generated by $u\zeta^{-1}/r^3$ or $z\sqrt{-1}/r^3$, it becomes necessary in certain instances (cases involving address substitution in cycles 3 and 5) to specify j_n before obtaining the first partition of the characteristic. It is impossible here to transmit the number of the characteristic of P, Q, R,... directly into the cell $<j_n>$; during operation of the partition block, the contents of this cell change. However, if we send the number of the characteristic into the cell $<j_2>$, it is automatically transmitted to the cell $<j_n>$ in the operation of the partition block.

These remarks indicate what changes must be introduced into the logical diagram of the preceding block in order to obtain the block described in the present section. /674



We will list the descriptions only of those operators whose function differs from the function of the corresponding operators of the preceding block.

T_2 transmits to the operators VIII-16, I-16 commands for return to the present block; it transmits to the reference cells constants for correction of the limits of variation of the parameter j_1 (the operator T_2 performs a similar function in the preceding block).

A_3 determines the order of the characteristic λ.

P_4 tests the following condition: The order of the characteristic is greater than or equal to n + 1, where n is the number of the row of the expansion. If this condition is not fulfilled, control is transferred to the master executive routine, which executes a transition to the third stage of operation.

T_{11} transmits into the cell $\langle j_2 \rangle$ the characteristic of whichever of the quantities P, Q, R,... corresponds to the term of the expansion with number $f = 1 + i_1 + i_2 + i_3$.

T_{12} transmits the characteristic λ into the cell D_1 of the partition block.

T_{13} transmits into the cell $\langle j_2 \rangle$ the characteristic of whichever of the quantities P, Q, R,... corresponds to the term of the expansion with number $f = 1 + i_1 + i_2 + i_3$.

$E_{1_{18}}$ reverts to the subprogram - to the operators A_{10} - T_{11}.

$E_{1_{21}}$ reverts to the subprogram - to the operators O_8 - T_{11}.

A_{27} replaces i by $i + i_1 + \tilde{i}_2$.

We conclude the present article by considering one further problem, which still has to be cleared up. How can the present program be used to compute the terms generated by the first row of the expansion of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$? This is accomplished by introducing the fictitious inequality

$$\sum_i [\text{illegible}] \quad \text{where } [\text{illegible}] \begin{cases} = 0 \text{ for } [\text{illegible}] \neq 0, \\ = 1 \text{ for } i = 0. \end{cases}$$

Since multiplication by this inequality is equivalent to multiplication by 1, it can be used to represent the terms of the first row in a form analogous to that of the terms in the other rows. In order to execute multiplication by this inequality according to the general rule, its coefficients and characteristic are injected into machine memory with $i_1 = i_2 = i_3 = i_4 = 0$. The number of the initial cell of the inequality is written in the second address of the second cell of this characteristic. For the terms of the first row of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$, the characteristics λ/e'^h can only be subjected to one partition $\frac{\lambda}{e'^h} = \tilde{1} \cdot \frac{\lambda}{e'^h}$ where $\tilde{1}$ is the characteristic of the fictitious inequality. If the characteristic

$\tilde{1}$ is inserted in the block of characteristics ahead of the characteristic 1, its order number will be equal to 0 and the limits of variation of j_1 in the case of the first row of $\partial\Omega/\partial s$ or $\partial\Omega/\partial z$ must be chosen such that j_1 will assume the unique value

$$j_1 = J = 0.$$

This operation is carried out by a special operator not indicated in the diagram of the partition block (coming after Φ_3). The order number j of the characteristic of the second factor λ/e'^h can be determined in the usual manner. /675

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